

Approximation of stationary solutions to SDEs driven by multiplicative fractional noise

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Abstract

In a previous paper, we studied the ergodic properties of an Euler scheme of a stochastic differential equation with a Gaussian additive noise in order to approximate the stationary regime of such equation. We now consider the case of multiplicative noise when the Gaussian process is a fractional Brownian Motion with Hurst parameter $H > 1/2$ and obtain some (functional) convergences properties of some empirical measures of the Euler scheme to the stationary solutions of such SDEs.

Keywords: stochastic differential equation; fractional Brownian motion; stationary process; Euler scheme.

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1 Introduction

Stochastic Differential Equations (SDEs) driven by a fractional Brownian motion (fBm) have been introduced to model random evolution phenomena whose noise has long range dependence properties. Indeed, beyond the historical motivations in hydrology and telecommunication for the use of fBm (highlighted e.g in [21]), recent applications of dynamical systems driven by this process include challenging issues in Finance [11], Biotechnology [24] or Biophysics [15, 16]. As a consequence, SDEs driven by fBm have been widely studied in a finite-time horizon during the last decades, and the reader is referred to [23, 4] for nice overviews on this topic.

In a somehow different direction, the study of the long-time behavior (under some stability properties) for fractional SDEs has been developed by Hairer (see [12, 13]) and Hairer and Ohashi [14], who built a way to define stationary solutions of these a priori non-Markov processes and to extend some of the tools of the Markovian theory to this setting. See also [1, 5, 10] for another setting called Random dynamical systems. The current article fits into this global aim, and starts from the following observation: the knowledge of the stationary regime being important for applications and essentially inaccessible in an explicit form, we propose to build and to study a procedure for its approximation in the

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case of SDEs driven by fBm with a Hurst parameter $H > 1/2$. This paper is following a similar previous work for SDEs driven by more general noises but in the specific additive case (see [3]).

More precisely, we deal with an \mathbb{R}^d -valued process $(X_t)_{t \geq 0}$ solution to the following SDE

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t^H \quad (1)$$

where $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \rightarrow \mathbb{M}_{d,q}$ are (at least) continuous functions, and where $\mathbb{M}_{d,q}$ is the set of $d \times q$ real matrices. In (1), $(B_t^H)_{t \geq 0}$ is a q -dimensional H -fBm and for the sake of simplicity we assume $\frac{1}{2} < H < 1$, which allows in particular to invoke Young integration techniques in order to define stochastic integrals with respect to B . Compared to [3] we handle here a fairly general diffusion coefficient σ , instead of the constant one considered previously. Classically the noise is called multiplicative in this setting, whereas it is called additive when σ is constant.

Under some Hölder regularity assumptions on the coefficients (see Section 2 for details), (strong) existence and uniqueness hold for the solution to (1) starting from $x_0 \in \mathbb{R}^d$. Classically for any stochastic differential equation, a natural question arises: if we assume that some Lyapunov assumptions hold on the drift term, does it imply that $(X_t)_{t \geq 0}$ has some convergence properties (in distribution or in a pathwise sense) to a steady state when $t \rightarrow +\infty$?

This question implies in particular to define rigorously a concept of steady state. For equation (1), this work has been done in [14]: using the fact that, owing to the Mandelbrot representation, the evolution of the fBm can be represented through a Feller transition on a functional space \mathcal{S} , the authors show that a solution to (1) can be built as the first coordinate of an homogeneous Markov process on the product space $\mathbb{R}^d \times \mathcal{S}$. As a consequence, stationary regimes associated with (1) can be naturally defined as the first projection of invariant measures of this Markov process. Furthermore, the authors of [14] develop some specific theory on strong Feller and irreducibility properties to prove uniqueness of invariant measures in this context.

In the current article, our aim is to propose a way to approximate numerically the stationary solutions to equation (1). To this end, we study some empirical occupation measures corresponding to an Euler type approximation of (1) with step $\gamma > 0$. We show that, under some Lyapunov assumptions, a (pathwise) convergence to the stationary solution of the discretized equation (denoted by ν^γ) holds and that, when $\gamma \rightarrow 0^+$, ν^γ converges in turn to the stationary solution of (1). This approach is the same as in [3]. However, the introduction of multiplicative noise has some important consequences on the techniques for proving the long-time stability of the Euler scheme. In particular, the main difficulty is to show that the long-time control of the dynamical system can be achieved independently of γ . In [3], this problem has been solved with the help of explicit computations for an Ornstein-Uhlenbeck type process. Because the noise is multiplicative the computations of [3] are not feasible anymore and we use specific tools to obtain uniform controls of discretized integrals with respect to the fBm. Before going more precisely to the heart of the matter, let us mention that the numerical approximation of the stationary regime by occupation measures of Euler schemes is a classical problem in a Markov setting including diffusions and Lévy driven SDEs (see *e.g.* [28, 17, 18, 19, 25, 26]).

2 Framework and main results

For some fixed $H \in (\frac{1}{2}, 1)$, we consider $(\Omega, \mathcal{F}, \mathbb{P})$ the canonical probability space associated with the fractional Brownian motion indexed by \mathbb{R} with Hurst parameter H . That is,

$\Omega = \mathcal{C}_0(\mathbb{R})$ is the Banach space of continuous functions vanishing at 0 equipped with the supremum norm, \mathcal{F} is the Borel sigma-algebra and \mathbb{P} is the unique probability measure on Ω such that the canonical process $B = \{B_t = (B_t^1, \dots, B_t^q), t \in \mathbb{R}\}$ is a fractional Brownian motion with Hurst parameter H . In this context, let us recall that B is a q -dimensional centered Gaussian process such that $B_0 = 0$, whose coordinates are independent and satisfy

$$\mathbb{E} \left[\left(B_t^j - B_s^j \right)^2 \right] = |t - s|^{2H}, \quad \text{for } s, t \in \mathbb{R}. \quad (2)$$

In particular it can be shown, by a standard application of Kolmogorov's criterion, that B admits a continuous version whose paths are θ -Hölder continuous for any $\theta < H$.

Let us be more specific about the definition of Hölder spaces of continuous functions. Namely, our driving process B lies into a space \mathcal{C}^θ defined as follows: we denote by $\mathcal{C}^\theta(\mathbb{R}_+, \mathbb{R}^d)$ the set of functions $f : \mathbb{R}_+ \rightarrow \mathbb{R}^d$ such that

$$\forall T > 0, \quad \|f\|_{\theta, T} = \sup_{0 \leq s < t \leq T} \frac{|f(t) - f(s)|}{(t - s)^\theta} < +\infty,$$

where the Euclidean norm is denoted by $|\cdot|$. We recall that $\mathcal{C}^\theta(\mathbb{R}_+, \mathbb{R}^d)$ can be made into a non-separable complete metric space, whenever endowed with the distance δ_θ defined by

$$\delta_\theta(f, g) = \sum_{N \in \mathbb{N}} 2^{-N} \left(1 \wedge \left(\sup_{0 \leq t \leq N} \|f(t) - g(t)\| + \|f - g\|_{\theta, N} \right) \right),$$

where $x \wedge y = \min(x, y) \forall x, y \in \mathbb{R}$. However, since separable spaces are crucial for convergence in law issues, we will work in fact with a smaller space $\bar{\mathcal{C}}^\theta(\mathbb{R}_+, \mathbb{R}^d)$: we say that a function f in $\mathcal{C}^\theta(\mathbb{R}_+, \mathbb{R}^d)$ belongs to $\bar{\mathcal{C}}^\theta(\mathbb{R}_+, \mathbb{R}^d)$ if

$$\forall T > 0, \quad \omega_{\theta, T}(f, \delta) := \sup_{0 \leq s < t < T, 0 \leq |t-s| \leq \delta} \frac{|f(t) - f(s)|}{|t - s|^\theta} \xrightarrow{\delta \rightarrow 0} 0.$$

$\bar{\mathcal{C}}^\theta(\mathbb{R}_+, \mathbb{R}^d)$ is a closed separable subspace of $\mathcal{C}^\theta(\mathbb{R}_+, \mathbb{R}^d)$.

We recall now some results on the existence and uniqueness of the solutions of the stochastic differential equation (1) starting from a deterministic point a . Let us suppose that b is Lipschitz continuous and that σ is $(1 + \alpha)$ -Lipschitz with $\alpha > \frac{1}{H} - 1$. We recall that for $0 < \alpha < 1$, σ is $(1 + \alpha)$ -Lipschitz on \mathbb{R}^d if it is a C^1 function and if the following norm is finite:

$$\|\sigma\|_{1+\alpha} = \sup_{x \in \mathbb{R}^d} \|D\sigma(x)\| + \sup_{x, y \in \mathbb{R}^d, |x-y| \leq \delta} \frac{|D\sigma(x) - D\sigma(y)|}{|x - y|^\alpha}. \quad (3)$$

Then for any deterministic function $B \in \mathcal{C}^\theta(\mathbb{R}_+, \mathbb{R}^q)$ with $\theta > \frac{1}{2}$, any $x_0 \in \mathbb{R}^d$ and any $(1 + \alpha)$ -Lipschitz map σ , there exists a unique solution denoted by $\Phi(x_0, B)$ in $\mathcal{C}^\theta(\mathbb{R}_+, \mathbb{R}^d)$ of

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \quad X_0 = x_0. \quad (4)$$

See [4, 20] for proofs and discussions of this kind of deterministic ordinary differential equations. Please note that by definition

$$\Phi(x_0, B)_t = x_0 + \int_0^t b(\Phi(x_0, B)_s)ds + \int_0^t \sigma(\Phi(x_0, B)_s)dB_s,$$

where the integrals are Riemman Stieljes integrals. Moreover the so-called Itô map Φ is continuous from $\mathbb{R}^d \times \mathcal{C}^\theta(\mathbb{R}_+, \mathbb{R}^q)$ into $\mathcal{C}^\theta(\mathbb{R}_+, \mathbb{R}^d)$ (see Proposition 47 in [4]). Actually

the solution of the stochastic differential equation (1) such that $X_0 = x_0$ is nothing but $\Phi(x_0, B^H(\omega))$, where the fractional Brownian motion B^H has been first sampled.

Let us now introduce a long-time stability assumption **(C)** concerning the stochastic differential equation (1) where $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a continuous function.

Let $\mathcal{EQ}(\mathbb{R}^d)$ denote the set of *Essentially Quadratic* functions, that is \mathcal{C}^2 -functions $V : \mathbb{R}^d \rightarrow (0, \infty)$ such that

$$\liminf_{|x| \rightarrow +\infty} \frac{V(x)}{|x|^2} > 0, \quad |\nabla V| \leq C\sqrt{V} \quad \text{and} \quad D^2V \text{ is bounded.}$$

Note that any element $V \in \mathcal{EQ}(\mathbb{R}^d)$ is continuous, and thus attains its positive minimum $\underline{v} > 0$ so that, for any $A, r > 0$, there exists a real constant $C_{A,r}$ such that $A + V^r \leq C_{A,r} V^r$.

(C): The map σ is assumed to be a bounded Lipschitz continuous function. Moreover we suppose that there exists $V \in \mathcal{EQ}(\mathbb{R}^d)$ such that

$$(i) \text{ for a given } C > 0 \text{ we have } |b(x)|^2 \leq CV(x) \quad \forall x \in \mathbb{R}^d.$$

$$(ii) \text{ and for } \beta \in \mathbb{R} \text{ and } \alpha > 0, \text{ the following relation holds:}$$

$$\forall x \in \mathbb{R}^d, \quad \langle \nabla V(x), b(x) \rangle \leq \beta - \alpha V(x).$$

We now turn to the definition of a stationary solution of (1).

DEFINITION 1. For $1/2 < \theta < H < 1$, a stationary solution ν to (1) is a probability measure on $\mathcal{C}^\theta(\mathbb{R}_+, \mathbb{R}^d)$, which is the first projection of a probability measure μ on $\mathcal{C}^\theta(\mathbb{R}_+, \mathbb{R}^d) \times \mathcal{C}^\theta(\mathbb{R}, \mathbb{R}^q)$, satisfying the following conditions. Let us denote by (X_t, B_t) the coordinate process on $\mathcal{C}^\theta(\mathbb{R}_+, \mathbb{R}^d) \times \mathcal{C}^\theta(\mathbb{R}, \mathbb{R}^q)$. The projection of μ on $\mathcal{C}^\theta(\mathbb{R}, \mathbb{R}^q)$ is the law of a H -fractional Brownian motion indexed by \mathbb{R} . For any $T > 0$, we consider μ_T the restriction of μ on $\mathcal{C}^\theta([0, T], \mathbb{R}^d) \times \mathcal{C}^\theta([0, T], \mathbb{R}^q)$. Then we assume that

$$X = \Phi(X_0, (B_t^H)_{t \geq 0}), \quad \mu_T \text{ almost surely} \quad (5)$$

and that the distribution of $(X_t)_{t \geq 0}$ is strictly stationary under μ . A stationary solution is called adapted, if for $0 \leq s \leq t$ the processes $(X_s)_{0 \leq s \leq t}$ and $(B^H(s))_{s \geq t}$ are conditionally independent given $(B^H(s))_{s \leq t}$.

This definition is very similar to the definition in [14] that comes from Stochastic Dynamical Systems.

Let γ be a positive number, we will now discretize equation (1) as follows:

$$Y_t^\gamma = Y_{n\gamma}^\gamma + (t - n\gamma)b(Y_{n\gamma}^\gamma) + \sigma(Y_{n\gamma}^\gamma)(B_t^H - B_{n\gamma}^H) \quad \forall t \in [n\gamma, (n+1)\gamma). \quad (6)$$

We set

$$\underline{t}_\gamma = \max\{\gamma k, \gamma k \leq t, k \in \mathbb{N}\}.$$

In fact, we will usually write \underline{t} instead of \underline{t}_γ in the sequel. The discretization of (1) can also be introduced with the following discretization $\Phi^\gamma : \mathbb{R} \times \mathcal{C}^\theta(\mathbb{R}_+, \mathbb{R}^q) \mapsto \mathcal{C}^\theta(\mathbb{R}_+, \mathbb{R}^d)$ of the Itô map :

$$\Phi^\gamma(x_0, B)_t := x_0 + \int_0^t b(\Phi^\gamma(x_0, B)_{\underline{s}_\gamma}) ds + \int_0^t \sigma(\Phi^\gamma(x_0, B)_{\underline{s}_\gamma}) dB_s. \quad (7)$$

Please note that the definition of Φ^γ does not involve any Riemann integration but only finite sums and that

$$Y^\gamma = \Phi^\gamma(Y_0^\gamma, (B_t^H)_{t \geq 0})_t. \quad (8)$$

Then, we call adapted solution of (6) a distribution ν^γ on $\mathcal{C}^\theta(\mathbb{R}_+, \mathbb{R}^d)$ defined as in Definition 1 replacing (5) by (8). We will say that ν^γ is stationary if it is invariant by the shift maps $(\theta_{k\gamma})_{k \in \mathbb{N}}$, where $\theta_t : \mathcal{C}^\theta(\mathbb{R}_+, \mathbb{R}^d) \rightarrow \mathcal{C}^\theta(\mathbb{R}_+, \mathbb{R}^d)$ is such that $(\theta_t(\omega))_s = \omega_{t+s}$, for $t \geq 0$. Note that in this definition, there is a slight abuse of language since we do not require the invariance by the shift maps θ_t for every $t \geq 0$, but only when $t = k\gamma$, $k \in \mathbb{N}$.

Let us introduce the following uniqueness assumption for ν^γ and ν :

(S $^\gamma$) ($\gamma \geq 0$): There is at most one adapted stationary solution to (1) (resp. to (8)) if $\gamma = 0$ (resp. if $\gamma > 0$).

For **(S 0)**, we refer to Theorem 1.1. of [14]. When $\gamma > 0$, we have the following proposition:

PROPOSITION 1. *Let $H \in (1/2, 1)$. Assume that $d = q$ and that b and σ are \mathcal{C}^2 -functions. Assume that σ is invertible and that $\sup_{x \in \mathbb{R}^d} \sigma^{-1}(x) < +\infty$. Then, **(S $^\gamma$)** holds for every $\gamma > 0$.*

The proof, which is an application of [13], is done in the appendix.

Let us now focus on the construction of the approximation: we denote by $(\bar{X}_t^\gamma)_{t \geq 0}$ the continuous-time Euler scheme defined by: $\bar{X}_0^\gamma = x \in \mathbb{R}^d$ and for every $n \geq 0$

$$\bar{X}_t^\gamma = \bar{X}_{n\gamma}^\gamma + (t - n\gamma)b(\bar{X}_{n\gamma}^\gamma) + \sigma(\bar{X}_{n\gamma}^\gamma)(B_t^H - B_{n\gamma}^H) \quad \forall t \in [n\gamma, (n+1)\gamma). \quad (9)$$

The process $(\bar{X}_t^\gamma)_{t \geq 0}$ is a solution to (6) such that $\bar{X}_0^\gamma = x$. In order to alleviate the notations and, when it is not confusing, we will usually write \bar{X}_t instead of \bar{X}_t^γ . Now, we define a sequence of random probability measures $(\mathcal{P}^{(n)}(\omega, d\alpha))_{n \geq 1}$ on $\bar{\mathcal{C}}^\theta(\mathbb{R}_+, \mathbb{R}^d)$ with $\theta < H$ by

$$\mathcal{P}^{(n,\gamma)}(\omega, d\alpha) = \frac{1}{n} \sum_{k=1}^n \delta_{\bar{X}_{\gamma(k-1)+}^\gamma(\omega)}(d\alpha)$$

where δ denotes the Dirac measure and for every $s \geq 0$, $\bar{X}_{s+}^\gamma := (\bar{X}_{s+t}^\gamma)_{t \geq 0}$ denotes the s -shifted process.

For $t \geq 0$, the sequence $(\mathcal{P}_t^{(n)}(\omega, dy))_{n \geq 1}$ of “marginal” empirical measures at time t on \mathbb{R}^d is defined by

$$\mathcal{P}_t^{(n,\gamma)}(\omega, dy) = \frac{1}{n} \sum_{k=1}^n \delta_{\bar{X}_{\gamma(k-1)+t}^\gamma(\omega)}(dy).$$

We are now able to state the main theorem of this article:

THEOREM 1. *Let $1/2 < \theta < H < 1$. Assume **(C)** and **(S $^\gamma$)** for every $\gamma > 0$. Then,*
(i) There exists $\gamma_0 > 0$ such that for every $\gamma \in (0, \gamma_0)$,

$$\mathcal{P}^{(n,\gamma)}(\omega, d\alpha) \xrightarrow{(\bar{\mathcal{C}}^\theta(\mathbb{R}_+, \mathbb{R}^d))} \nu^\gamma(d\alpha) \quad \text{a.s. where } n \rightarrow +\infty$$

and where ν^γ denotes the unique adapted stationary solution to (6).

*(ii) Moreover, if b is Lipschitz continuous, σ is $(1 + \alpha)$ -Lipschitz with $\alpha > \frac{1}{H} - 1$ and if **(S 0)** holds, then,*

$$\lim_{\gamma \rightarrow 0} \lim_{n \rightarrow +\infty} \mathcal{P}^{(n,\gamma)}(\omega, d\alpha) = \nu(d\alpha) \quad \text{a.s.}$$

where the convergence is again for the weak topology induced by $\bar{\mathcal{C}}^\theta(\mathbb{R}_+, \mathbb{R}^d)$ and ν denotes the unique adapted stationary solution to (1).

REMARK 1. Note that some extensions can be deduced from the proof of this theorem. First, when uniqueness fails for the stationary solutions, the preceding result is replaced by

THEOREM 2. 1. Assume **(C)**. Then, there exists $\gamma_0 > 0$ such that for every $\gamma \in (0, \gamma_0)$, $(\mathcal{P}^{(n,\gamma)}(\omega, d\alpha))_{n \geq 1}$ is a.s. tight on $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$. Furthermore, every weak limit is a stationary adapted solution of (6).
2. Assume **(C)** and set

$$\mathcal{U}^{\infty,\gamma}(\omega) := \{\text{weak limits of } (\mathcal{P}^{(n,\gamma)}(\omega, d\alpha))\}.$$

Then, there exists $\gamma_1 \in (0, \gamma_0)$ such that $(\mathcal{U}^{\infty,\gamma}(\omega))_{\gamma \leq \gamma_1}$ is a.s. relatively compact for the uniform convergence topology on compact sets and any weak limit when $\gamma \rightarrow 0$ of $(\mathcal{U}^{\infty,\gamma}(\omega))_{\gamma \leq \gamma_1}$ is an adapted stationary solution of (1).

REMARK 2. From the very definition of weak convergence, the preceding assertions imply that the convergence of $(\mathcal{P}^{(n,\gamma)}(\omega, d\alpha))_{n,\gamma}$ holds for bounded continuous functionals $F : \bar{C}^\theta(\mathbb{R}_+, \mathbb{R}^d) \rightarrow \mathbb{R}$. In fact, this convergence can be extended to some non-bounded continuous functionals $F : \bar{C}^\theta([0, T], \mathbb{R}^d) \rightarrow \mathbb{R}$. Actually, setting $G(\alpha) = \sup_{t \in [0, T]} V(\alpha_t)$, we easily deduce from inequality (12) of Proposition 3 and Proposition 4 that

$$\sup_{\gamma \leq \gamma_0} \limsup_{n \rightarrow +\infty} \mathcal{P}^{(n,\gamma)}(\omega, G^p) < +\infty \quad a.s.$$

for every $p > 0$. By an uniform integrability argument, it follows

PROPOSITION 2. The convergence properties of $(\mathcal{P}^{(n,\gamma)}(\omega, d\alpha))$ extend to continuous functionals $F : \bar{C}^\theta([0, T], \mathbb{R}^d) \rightarrow \mathbb{R}$ such that for every $\alpha \in \bar{C}^\theta([0, T], \mathbb{R}^d)$,

$$|F(\alpha_t, 0 \leq t \leq T)| \leq C \sup_{t \in [0, T]} V^p(\alpha_t)$$

with $T > 0$ and $p > 0$.

REMARK 3. A third natural extension of Theorem 1 consists in handling the case of an irregular fractional Brownian motion B with Hurst index $1/4 < H < 1/2$. This extension is presumably within the reach of our technology on differential systems driven by fBm, but requires a huge amount of technical elaboration. Indeed, to start with, equation (1) has to be defined thanks to rough paths techniques whenever $H < 1/2$, and we refer to [9] for a complete account on rough differential equations driven by Gaussian processes in general and fractional Brownian motion in particular. More importantly, as it will be observed in the next sections, our main result heavily relies on some thorough estimates performed on the discretized version (6) of equation (1). When $H > 1/2$ this discretization procedure is based on an Euler type scheme, but the case $H < 1/2$ involves the introduction of some Levy area correction terms of Milstein type (see [6]) or products of increments of B if one desires to deal with an implementable numerical scheme (cf. [7]). This new setting has tremendous effects on the proof of Propositions 3 and 4. For sake of conciseness, we have thus decided to stick to the case $H > 1/2$, and defer the rough case to a subsequent publication.

The sequel of the paper is built as follows. The three next sections are devoted to the proof of Theorem 1. In Section 3, we prove some preliminary results for the long-time stability of $(\mathcal{P}^{(n,\gamma)}(\omega, d\alpha))_n$, when $\gamma > 0$. It is important to note that the controls established in this section are independent of γ in order to obtain in the sequel a long-time control that does not explode when $\gamma \rightarrow 0$. Then, in Section 4, we obtain some

tightness properties for $(\mathcal{P}^{(n,\gamma)}(\omega, d\alpha))$ (in n and γ) and, in Section 5, we prove that the weak limits of this sequence are adapted stationary solutions. Eventually, in Section 6, we test numerically our algorithm for the approximation of the invariant distribution of a particular fractional SDE.

Note that in the proofs below, every non-explicit constant is denoted by C and may change from line to line.

3 Evolution control of (\bar{X}_t^γ) in a finite horizon

The main aim of this part is to obtain a finite-time control of $V(\bar{X}_T^\gamma)$ in terms of $V(\bar{X}_0^\gamma)$ which is somewhat contracting and which is independent of γ . This is the purpose of the first part of Proposition 3 below. In order to obtain some functional convergence results, we state in the second part a result about the finite-time control of the Hölder semi-norm of \bar{X}^γ .

PROPOSITION 3. *Let $T > 0$. Assume (C). Then,*

(i) *For every $p \geq 1$, there exist $\gamma_0 > 0$, $\rho \in (0, 1)$ and a polynomial function $P_{p,\theta} : \mathbb{R} \rightarrow \mathbb{R}$ such that for every $\gamma \in (0, \gamma_0]$,*

$$V^p(\bar{X}_T^\gamma) \leq \rho V^p(x) + P_{p,\theta}(\|B^H\|_{\theta,T}) + CQ_\gamma^p(B_t^H, 0 \leq t \leq T), \quad (10)$$

where Q_γ is defined for every $(w(t))_{t \in [0,T]}$ by

$$Q_\gamma((w(t))_{t \in [0,T]}) = \sum_{k=1}^{\lfloor \frac{T}{\gamma} \rfloor} |w(k\gamma) - w((k-1)\gamma)|^2. \quad (11)$$

Furthermore,

$$\sup_{t \in [0,T]} V^p(\bar{X}_t^\gamma) \leq C(V^p(x) + P_{p,\theta}(\|B^H\|_{\theta,T}) + Q_\gamma^p(B_t^H, 0 \leq t \leq T)). \quad (12)$$

(ii) *For every $\theta \in (\frac{1}{2}, H)$, $T > 0$, and $\gamma \in (0, \gamma_0]$*

$$\sup_{0 \leq s < t \leq T} \frac{|\bar{X}_t^\gamma - \bar{X}_s^\gamma|}{(t-s)^\theta} \leq C_T(V(x) + \tilde{P}_\theta(\|B^H\|_{\theta,T}) + Q_\gamma(B_t^H, 0 \leq t \leq T))$$

where \tilde{P} is another real valued polynomial function.

The proof of this result is achieved in Subsection 3.2. Before, we focus in Subsection 3.1 on the control of increments of some discretized equations with non-bounded coefficients driven by B^H .

3.1 Technical Lemmas

Let us recall that, for every $t \geq 0$, $\underline{t}_\gamma = \gamma \max\{k \in \mathbb{N}, \gamma k \leq t\}$. In the sequel, we will usually write \underline{t} instead of \underline{t}_γ .

In the following lemmas, we will use the following notation: for any element $(x(t))_{t \geq 0}$ of $\mathcal{C}(\mathbb{R}_+, \mathbb{R}^d)$ and $T > 0$, $\theta > 0$, $\gamma > 0$, we define

$$\|x\|_{\theta,\gamma}^{s,t} = \sup_{s \leq u \leq v \leq t} \frac{|x(\underline{v}_\gamma) - x(\underline{u}_\gamma)|}{(\underline{v}_\gamma - \underline{u}_\gamma)^\theta},$$

where we set by convention $\frac{0}{0} = 0$.

LEMMA 1. Assume that b is a sublinear function, i.e. that there exists $C > 0$ such that for every $x \in \mathbb{R}^d$, $|b(x)| \leq C(1 + |x|)$. Then, for every $T > 0$, there exists a constant $C > 0$ such that for every $s, t \in [0, T]$ with $s \leq t$, for every $\gamma > 0$, for every $\theta \in (0, H)$

$$|\bar{X}_{\underline{t}}^\gamma| \leq \left(|\bar{X}_{\underline{s}}^\gamma| + C(\underline{t} - \underline{s}) + \|\bar{Z}^\gamma\|_{\theta, \gamma}^{s, t} (\underline{t} - \underline{s})^\theta \right) \exp(C(\underline{t} - \underline{s}))$$

where

$$\bar{Z}_t^\gamma = \int_0^t \sigma(\bar{X}_s^\gamma) dB_s^H.$$

Proof. First, from the very definition of $(\bar{X}_t^\gamma)_{t \geq 0}$, we have for every $s, t \in [0, T]$ with $s \leq t$:

$$\bar{X}_{\underline{t}}^\gamma = \bar{X}_{\underline{s}}^\gamma + \int_{\underline{s}}^{\underline{t}} b(\bar{X}_u^\gamma) du + \bar{Z}_{\underline{t}}^\gamma - \bar{Z}_{\underline{s}}^\gamma. \quad (13)$$

The function b being sublinear, we deduce that

$$|\bar{X}_{\underline{t}}^\gamma| = |\bar{X}_{\underline{s}}^\gamma| + \|\bar{Z}^\gamma\|_{\theta, \gamma}^{s, t} (\underline{t} - \underline{s})^\theta + C \int_{\underline{s}}^{\underline{t}} (1 + |\bar{X}_u^\gamma|) du.$$

Setting $g_s(v) = |\bar{X}_{\underline{s}+v}^\gamma|$, it follows that for every $v \in [0, \underline{t} - \underline{s}]$,

$$g_s(v) \leq a + C \int_0^v g_s(u) du$$

with $a = |\bar{X}_{\underline{s}}^\gamma| + \|\bar{Z}^\gamma\|_{\theta, \gamma}^{s, t} (\underline{t} - \underline{s})^\theta + C(\underline{t} - \underline{s})$. The result follows from Gronwall's lemma. \square

The control of B^H -integrals is usually based on the so-called sewing Lemma (see e.g. [4, 8]) which leads to a comparison of $\int_s^t f(x_u) dB_u^H$ with $f(x_t)(B_t^H - B_s^H)$. The following lemma can be viewed as a discretized version of such results:

LEMMA 2. Let $\gamma_0 > 0$ and $(f_\gamma)_{\gamma \in (0, \gamma_0]}$ be a family of functions from $\mathbb{R}_+ \times \mathbb{R}^d$ to $\mathbb{M}_{d, q}$ such that there exists $r \geq 0$ such that for every $T > 0$, there exists $C_T > 0$ such that $\forall \gamma \in (0, \gamma_0]$,

$$\forall (s, x), (t, y) \in [0, T] \times \mathbb{R}^d, \quad \|f_\gamma(t, y) - f_\gamma(s, x)\| \leq C_T(1 + |x|^r + |y|^r)(|t - s| + |y - x|). \quad (14)$$

Let $(\bar{H}_t^\gamma)_{t \geq 0}$ be defined by

$$\forall t \geq 0, \quad \bar{H}_t^\gamma = \int_0^t f_\gamma(\underline{s}, \bar{X}_{\underline{s}}^\gamma) dB_s^H.$$

Then, for every $\theta \in (\frac{1}{2}, H)$, for every $T > 0$, there exists $\tilde{C}_T > 0$ such that for every $\gamma \in (0, \gamma_0]$, for every $0 \leq s \leq t \leq T$,

$$|\bar{H}_{\underline{t}}^\gamma - \bar{H}_{\underline{s}}^\gamma - f_\gamma(\underline{s}, \bar{X}_{\underline{s}}^\gamma)(B_{\underline{t}}^H - B_{\underline{s}}^H)| \leq \tilde{C}_T(\underline{t} - \underline{s})^{2\theta} ((1 + |\bar{X}_{\underline{s}}^\gamma|^{r+1} + (\|\bar{Z}^\gamma\|_{\theta, \gamma}^{s, t-\gamma})^{r+1}) \|B^H\|_{\theta, T}).$$

Proof. Let $s, t \in [0, T]$ with $\underline{s} = \gamma i$ and $\underline{t} = \gamma j$ with $i < j$. We build the sequences $(\tau_l^{(k)})_{l=0}^{2^k}$ by: $\tau_0^{(0)} = \underline{s}$, $\tau_1^{(0)} = \underline{t}$ and for every $k \geq 1$, for every $l \in \{0, \dots, 2^k\}$:

$$\tau_l^{(k)} = \begin{cases} \tau_{\frac{l}{2}}^{(k-1)} & \text{if } l \text{ is even} \\ \gamma \left[\frac{1}{\gamma} (\tau_{\frac{l-1}{2}}^{(k-1)} + \tau_{\frac{l+1}{2}}^{(k-1)}) \right] & \text{if } l \text{ is odd.} \end{cases} \quad (15)$$

When l is odd, $\tau_l^{(k)}$ is in fact a discretization point which is located in the interval $[\tau_{\frac{l-1}{2}}^{(k-1)}, \tau_{\frac{l+1}{2}}^{(k-1)}]$. More precisely, it can also be defined as the largest discretization point on the left of the middle of this interval. It is important to remark that for k large enough, every discretization point between \underline{s} and \underline{t} is covered by one of the $\tau_l^{(k)}$'s, $l \in \{0, \dots, 2^k\}$. Then, we set

$$K = K_{ij} := \inf \left\{ k \in \mathbb{N}, \{ \gamma m, i \leq m \leq j \} \subset \{ \tau_l^k, 0 \leq l \leq 2^k \} \right\}.$$

Before going further, let us check that there exists a constant $C > 0$ ($C = 2$ is a suitable choice) such that for every $\gamma > 0$, for every $s, t \in [0, T]$

$$\forall k \in \{0, K\}, \quad \forall l \in \{1, \dots, 2^k\}, \quad \tau_l^{(k)} - \tau_{l-1}^{(k)} \leq C \frac{\underline{t} - \underline{s}}{2^k}. \quad (16)$$

If $K = 0$, this point is clearly satisfied. Assume now that $K \geq 1$. Denoting by $\bar{n}_{l,l+1}^{(k)}$ the number of discretization points (strictly) between $\tau_l^{(k)}$ and $\tau_{l+1}^{(k)}$, we can check by induction that

$$\forall k \in \{0, \dots, K\}, \quad \max\{\bar{n}_{l,l+1}^{(k)}, 0 \leq l \leq 2^k - 1\} - \min\{\bar{n}_{l,l+1}^{(k)}, 0 \leq l \leq 2^k - 1\} \leq 1.$$

In other words,

$$\forall k \in \{0, \dots, K\}, \quad \max\{\tau_{l+1}^{(k)} - \tau_l^{(k)}, 0 \leq l \leq 2^k - 1\} - \min\{\tau_{l+1}^{(k)} - \tau_l^{(k)}, 0 \leq l \leq 2^k - 1\} \leq \gamma.$$

Using that $\sum_{l=1}^{2^k} \tau_l^{(k)} - \tau_{l-1}^{(k)} = \underline{t} - \underline{s}$, we first deduce that

$$\max_{1 \leq l \leq 2^k} (\tau_l^{(k)} - \tau_{l-1}^{(k)}) \leq \frac{\underline{t} - \underline{s}}{2^k} + \gamma.$$

Owing to the definition of K , the second consequence is that $(\tau_l^{(K-1)})_l$ is (strictly) increasing and thus that $\min_{1 \leq l \leq 2^{K-1}} (\tau_l^{(K-1)} - \tau_{l-1}^{(K-1)}) \geq \gamma$. This implies that

$$\gamma \leq \frac{\underline{t} - \underline{s}}{2^{K-1}}$$

and (16) follows.

Now, let us return to the proof of the lemma. Using that at step K , the discretization steps are covered by the $\tau_l^{(K)}$'s, we obtain:

$$\bar{H}_{\underline{s}}^\gamma - \bar{H}_{\underline{s}}^\gamma = \sum_{l=0}^{2^K-1} f(\tau_l^{(K)}, \bar{X}_{\tau_l^{(K)}}^\gamma) (B_{\tau_{l+1}^{(K)}}^H - B_{\tau_l^{(K)}}^H).$$

Thanks to (15), we have for every $k \in \{0, \dots, K-1\}$ and $l \in \{0, \dots, 2^k - 1\}$:

$$B_{\tau_{l+1}^{(k)}}^H - B_{\tau_l^{(k)}}^H = (B_{\tau_{2l+2}^{(k+1)}}^H - B_{\tau_{2l+1}^{(k+1)}}^H) + (B_{\tau_{2l+1}^{(k+1)}}^H - B_{\tau_{2l}^{(k+1)}}^H).$$

Thus, setting

$$I_k = \sum_{l=0}^{2^k-1} f(\tau_l^{(K)}, \bar{X}_{\tau_l^{(K)}}^\gamma) (B_{\tau_{l+1}^{(k)}}^H - B_{\tau_l^{(k)}}^H),$$

we deduce that for every $k \in \{0, \dots, 2^{k-1}\}$,

$$I_k = I_{k+1} + \sum_{l=0}^{2^k-1} \varepsilon_\gamma(k, l)$$

with

$$\varepsilon_\gamma(k, l) = \left(f_\gamma(\tau_{2l}^{(k+1)}, \bar{X}_{\tau_{2l}^{(k+1)}}^\gamma) - f_\gamma(\tau_{2l+1}^{(k+1)}, \bar{X}_{\tau_{2l+1}^{(k+1)}}^\gamma) \right) \left(B_{\tau_{2l+2}}^H - B_{\tau_{2l+1}}^H \right).$$

Furthermore, $I_0 = f_\gamma(\underline{s}, \bar{X}_{\underline{s}}^\gamma)(B_{\underline{t}}^H - B_{\underline{s}}^H)$ and $I_K = \bar{H}_{\underline{t}}^\gamma - \bar{H}_{\underline{s}}^\gamma$. We then deduce that for every $\gamma > 0$,

$$\bar{H}_{\underline{t}}^\gamma - \bar{H}_{\underline{s}}^\gamma = f_\gamma(\underline{s}, \bar{X}_{\underline{s}}^\gamma)(B_{\underline{t}}^H - B_{\underline{s}}^H) - \sum_{k=0}^{K-1} \sum_{l=0}^{2^k-1} \varepsilon_\gamma(k, l). \quad (17)$$

Let us focus on the second part of the right-hand member of (17). Owing to the assumptions on f_γ , we have

$$|\varepsilon_\gamma(k, l)| \leq C \phi_{k,l} \left((\tau_{2l+1}^{(k+1)} - \tau_{2l}^{(k+1)}) + |\bar{X}_{\tau_{2l+1}^{(k+1)}}^\gamma - \bar{X}_{\tau_{2l}^{(k+1)}}^\gamma| \right) \left| B_{\tau_{2l+2}}^H - B_{\tau_{2l+1}}^H \right|$$

with $\phi_{k,l} = 1 + |\bar{X}_{\tau_{2l}^{(k+1)}}^\gamma|^r + |\bar{X}_{\tau_{2l+1}^{(k+1)}}^\gamma|^r$. Then,

$$|\varepsilon_\gamma(k, l)| \leq C \phi_{k,l} \left((\tau_{2l+1}^{(k+1)} - \tau_{2l}^{(k+1)}) + |\bar{X}_{\tau_{2l+1}^{(k+1)}}^\gamma - \bar{X}_{\tau_{2l}^{(k+1)}}^\gamma| \right) (\tau_{2l+2}^{(k+1)} - \tau_{2l+1}^{(k+1)})^\theta \|B^H\|_{\theta, T} \quad (18)$$

for every $\theta \in (\frac{1}{2}, H)$. Let us now focus on the control of $\phi_{k,l}$ and of $|\bar{X}_{\tau_{2l+1}^{(k+1)}}^\gamma - \bar{X}_{\tau_{2l}^{(k+1)}}^\gamma|$.

First, note that if $\tau_{2l+1}^{(k+1)} = \underline{t}$ then, $\tau_{2l+2}^{(k+1)} = \tau_{2l+1}^{(k+1)} = \underline{t}$ and it follows that in this case, $\varepsilon_\gamma(k, l) = 0$. In the sequel, we set $\mathcal{A}_t := \{(k, l), \tau_{2l+1}^{(k+1)} < \underline{t}\}$. By Lemma 1, for every $(k, l) \in \mathcal{A}_t$,

$$|\phi_{k,l}| \leq C_T \left(1 + |\bar{X}_{\underline{s}}^\gamma|^r + (\|\bar{Z}^\gamma\|_{\theta, \gamma}^{\underline{s}, \underline{t} - \gamma})^r \right).$$

Second, using the notation $n_\gamma(k, l)$ for the discretization index related to $\tau_l^{(k)}$ (i.e. $n_\gamma(k, l) = \gamma^{-1} \tau_l^{(k)}$), we can write

$$\bar{X}_{\tau_{2l+1}^{(k+1)}}^\gamma - \bar{X}_{\tau_{2l}^{(k+1)}}^\gamma = \sum_{v=n_\gamma(2l, k+1)}^{n_\gamma(2l+1, k+1)-1} \gamma b(\bar{X}_{v\gamma}^\gamma) + \bar{Z}_{\tau_{2l+1}^{(k+1)}}^\gamma - \bar{Z}_{\tau_{2l}^{(k+1)}}^\gamma.$$

Then, using that b is a sublinear function, we deduce from Lemma 1 that there exists $C_T > 0$ that does not depend on γ, l and k such that

$$\begin{aligned} \sum_{v=n_\gamma(2l, k+1)}^{n_\gamma(2l+1, k+1)-1} \gamma |b(\bar{X}_{v\gamma}^\gamma)| &\leq C_T \left(1 + |\bar{X}_{\underline{s}}^\gamma| + \|\bar{Z}^\gamma\|_{\theta, \gamma}^{\underline{s}, \tau_{2l+1}^{(k+1)} - \gamma} \right) (\tau_{2l+1}^{(k+1)} - \tau_{2l}^{(k+1)}) \\ &\leq C_T \left(1 + |\bar{X}_{\underline{s}}^\gamma| + \|\bar{Z}^\gamma\|_{\theta, \gamma}^{\underline{s}, \underline{t} - \gamma} \right) (\tau_{2l+2}^{(k+1)} - \tau_{2l}^{(k+1)}). \end{aligned}$$

Likewise, we have for every $(k, l) \in \mathcal{A}$,

$$|\bar{Z}_{\tau_{2l+1}^{(k+1)}}^\gamma - \bar{Z}_{\tau_{2l}^{(k+1)}}^\gamma| \leq \|\bar{Z}^\gamma\|_{\theta, \gamma}^{\underline{s}, \underline{t} - \gamma} (\tau_{2l+1}^{(k+1)} - \tau_{2l}^{(k+1)})^\theta.$$

Plugging the previous controls into (18) and using that $2\theta \leq 1 + \theta$, we obtain the existence of a constant $C_T > 0$ such that for every γ , l and k ,

$$|\varepsilon_\gamma(k, l)| \leq C_T \varphi_r(\bar{X}_s^\gamma, \|\bar{Z}^\gamma\|_{\theta, \gamma}^{\frac{s, t}{\theta} - \gamma}) \left(((1 + |\bar{X}_s^\gamma|)(\tau_{2l+2}^{(k+1)} - \tau_{2l}^{(k+1)})^{1+\theta} + \|\bar{Z}^\gamma\|_{\theta, \gamma}^{\frac{s, t}{\theta} - \gamma}(\tau_{2l+2}^{(k+1)} - \tau_{2l}^{(k+1)})^{2\theta}) \|B^H\|_{\theta, T} \right).$$

where

$$\varphi_r(x, r) = 1 + |x|^r + |z|^r.$$

Then, using (16) and the fact that

$$\sum_{l=0}^{2^k-1} \tau_{2l+2}^{(k+1)} - \tau_{2l}^{(k+1)} = \sum_{l=0}^{2^k-1} \tau_{l+1}^{(k)} - \tau_l^{(k)} = \underline{t} - \underline{s},$$

it follows that

$$\frac{\sum_{l=0}^{2^k-1} |\varepsilon_\gamma(k, l)|}{\varphi_r(\bar{X}_s^\gamma, \|\bar{Z}^\gamma\|_{\theta, \gamma}^{\frac{s, t}{\theta} - \gamma})} \leq C_T (\underline{t} - \underline{s}) \left((1 + |\bar{X}_s^\gamma|) \left(\frac{\underline{t} - \underline{s}}{2^{k+1}} \right)^\theta + \|\bar{Z}^\gamma\|_{\theta, \gamma}^{\frac{s, t}{\theta} - \gamma} \left(\frac{\underline{t} - \underline{s}}{2^{k+1}} \right)^{2\theta-1} \right) \|B^H\|_{\theta, T}.$$

Summing over k and using that $\sum_{k \geq 0} 2^{-(k+1)\theta}$ and $\sum_{k \geq 0} 2^{-(k+1)(2\theta-1)}$ are convergent, we deduce that

$$\begin{aligned} \sum_{k=0}^{K-1} \sum_{l=0}^{2^k-1} |\varepsilon_\gamma(k, l)| &\leq C_T \varphi_r(\bar{X}_s^\gamma, \|\bar{Z}^\gamma\|_{\theta, \gamma}^{\frac{s, t}{\theta} - \gamma}) \\ &\quad \times \left((1 + |\bar{X}_s^\gamma|)(\underline{t} - \underline{s})^{1+\theta} + \|\bar{Z}^\gamma\|_{\theta, \gamma}^{\frac{s, t}{\theta} - \gamma}(\underline{t} - \underline{s})^{2\theta} \right) \|B^H\|_{\theta, T}. \end{aligned} \quad (19)$$

A expansion of the right-hand side combined with elementary inequalities and the fact that $2\theta \leq 1 + \theta$ yields for every $0 \leq s \leq t \leq T$,

$$\sum_{k=0}^{K-1} \sum_{l=0}^{2^k-1} |\varepsilon_\gamma(k, l)| \leq C_T (\underline{t} - \underline{s})^{2\theta} \left(1 + |\bar{X}_s^\gamma|^{r+1} + \left(\|\bar{Z}^\gamma\|_{\theta, \gamma}^{\frac{s, t}{\theta} - \gamma} \right)^{1+r} \right) \|B^H\|_{\theta, T}$$

and the result follows from (17). \square

In the following lemma, we make use of Lemma 2 when $f_\gamma(t, x) = \sigma(x)$. In this particular case, we show below that we can deduce a control of the increments of \bar{Z}^γ on an interval with random but explicit length $\eta(\omega)$ (which does not depend on γ).

LEMMA 3. *Let γ_0 be a positive number. Assume that σ is a bounded Lipschitz continuous function. Then, for every $\theta \in (\frac{1}{2}, H)$, for every $T > 0$, there exists $C_T > 0$, there exists a positive random variable*

$$\eta(\omega) := \left(\frac{1}{2} [(C_T \|B^H(\omega)\|_{\theta, T})^{-1} \wedge 1] \right)^{\frac{1}{\theta}} \quad (20)$$

such that a.s for every $0 \leq s \leq t \leq T$ with $\underline{t} - \underline{s} \leq \eta$, for every $\gamma \in (0, \gamma_0)$

$$|\bar{Z}_t^\gamma - \bar{Z}_s^\gamma| \leq (\underline{t} - \underline{s})^\theta \left(\|\sigma\|_\infty + C_T (1 + |\bar{X}_s^\gamma|) \eta^\theta \right) \|B^H\|_{\theta, T}$$

where $\|\sigma\|_\infty = \sup_{x \in \mathbb{R}^d} \|\sigma(x)\|$.

Proof. For every $l \geq 0$, set $t_l = \underline{s} + \gamma l$ and $N_l = \|\bar{Z}^\gamma\|_{\theta, \gamma}^{\underline{s}, t_l}$. Owing to the definition of $\|\cdot\|_{\theta, \gamma}^{\underline{s}, t_l}$, we have

$$N_{l+1} \leq N_l \vee \sup_{i \leq l} \frac{|\bar{Z}_{t_{l+1}}^\gamma - \bar{Z}_{t_i}^\gamma|}{(t_{l+1} - t_i)^\theta}.$$

By Lemma 2 applied with $s = t_i$, $t = t_{l+1}$ and $f_\gamma(s, x) = \sigma(x)$ (and $r = 0$),

$$\frac{|\bar{Z}_{t_{l+1}}^\gamma - \bar{Z}_{t_i}^\gamma|}{(t_{l+1} - t_i)^\theta} \leq \left(\|\sigma\|_\infty + C_T(t_{l+1} - t_i)^\theta \left(1 + |\bar{X}_{t_i}^\gamma| + \|\bar{Z}^\gamma\|_{\theta, \gamma}^{\underline{s}, t_l} \right) \right) \|B^H\|_{\theta, T}.$$

By Lemma 1 and the fact that $t \rightarrow \|\bar{Z}^\gamma\|_{\theta, \gamma}^{\underline{s}, t}$ is nonincreasing, it follows that

$$\sup_{i \leq l} \frac{|\bar{Z}_{t_{l+1}}^\gamma - \bar{Z}_{t_i}^\gamma|}{(t_{l+1} - t_i)^\theta} \leq \left(\|\sigma\|_\infty + C_T \left((1 + |\bar{X}_{\underline{s}}^\gamma|)(t_{l+1} - \underline{s}) + \|\bar{Z}^\gamma\|_{\theta, \gamma}^{\underline{s}, t_l}(t_{l+1} - \underline{s})^\theta \right) \right) \|B^H\|_{\theta, T}.$$

Let ρ be a positive number. If $t_{l+1} - \underline{s} \leq \rho$, we obtain that

$$N_{l+1} \leq N_l \vee (\alpha_\rho + \beta_\rho N_l)$$

with

$$\alpha_\rho = \left(\|\sigma\|_\infty + C_T((1 + |\bar{X}_{\underline{s}}^\gamma|)\rho^\theta) \right) \|B^H\|_{\theta, T} \quad \text{and} \quad \beta_\rho = C_T \rho^\theta \|B^H\|_{\theta, T}.$$

Let us now set $\rho = \eta(\omega)$ where $\eta(\omega)$ is defined by (20). For this choice of ρ , we have $\beta_\eta \leq \frac{1}{2}$. Then, the interval $[0, \alpha_\eta/(1 - \beta_\eta)]$ being stable by the function $x \mapsto \alpha_\eta + \beta_\eta x$, we deduce that for every $l \in \mathbb{N}$ such that $t_{l+1} - \underline{s} \leq \eta(\omega)$,

$$N_l \leq \frac{\alpha_\eta}{2}.$$

The result follows. \square

3.2 Proof of Proposition 3

(i) We first prove (10) for $p = 1$. Set $\Delta_n = B_{\gamma n}^H - B_{\gamma(n-1)}^H$. Owing to the Taylor formula,

$$\begin{aligned} V(\bar{X}_{(n+1)\gamma}) &= V(\bar{X}_{n\gamma}) + \gamma \langle \nabla V(\bar{X}_{n\gamma}), b(\bar{X}_{n\gamma}) \rangle + \langle \nabla V(\bar{X}_{n\gamma}), \sigma(\bar{X}_{n\gamma}) \Delta_{n+1} \rangle \\ &\quad + \frac{1}{2} \sum_{i,j} \partial_{i,j}^2 V(\xi_{n+1})(\bar{X}_{(n+1)\gamma} - \bar{X}_{n\gamma})_i (\bar{X}_{(n+1)\gamma} - \bar{X}_{n\gamma})_j. \end{aligned}$$

where $\xi_{(n+1)} \in [\bar{X}_{n\gamma}, \bar{X}_{(n+1)\gamma}]$. Using Assumption (C) and the boundedness of D^2V and σ , we obtain

$$V(\bar{X}_{(n+1)\gamma}) \leq V(\bar{X}_{n\gamma}) + \gamma(\beta - \alpha V(\bar{X}_{n\gamma})) + A_1(n+1) + C(\gamma^2 V(\bar{X}_{n\gamma}) + |\Delta_{n+1}|^2). \quad (21)$$

where

$$A_1(n+1) = \langle \nabla V(\bar{X}_{n\gamma}), \sigma(\bar{X}_{n\gamma}) \Delta_{n+1} \rangle.$$

Set $\gamma_0 = \frac{\alpha}{2C}$. For every $\gamma \in (0, \gamma_0]$, for every $n \geq 0$, we have

$$V(\bar{X}_{(n+1)\gamma}) \leq V(\bar{X}_{n\gamma}) \left(1 - \frac{\alpha}{2} \gamma \right) + A_1(n+1) + (\beta\gamma + C|\Delta_{n+1}|^2).$$

Then, iterating the previous inequality yields for every s, t such that $s \leq t$,

$$V(\bar{X}_{\underline{t}}) \leq V(\bar{X}_{\underline{s}}) \left(1 - \frac{\alpha}{2}\gamma\right)^{\frac{t-s}{\gamma}} + \sum_{k=\frac{s}{\gamma}+1}^{\frac{t}{\gamma}} \left(1 - \frac{\alpha}{2}\gamma\right)^{\frac{t-s}{\gamma}-k} (A_1(k) + \beta\gamma + C|\Delta_k|^2).$$

Using that $\log(1+x) \leq x$ for every $x > -1$, we deduce that

$$V(\bar{X}_{\underline{t}}) \leq e^{-\frac{\alpha(\underline{t}-\underline{s})}{2}} (V(\bar{X}_{\underline{s}}) + |\bar{H}_{\underline{t}}^\gamma - \bar{H}_{\underline{s}}^\gamma|) + \sum_{k=\frac{s}{\gamma}+1}^{\frac{t}{\gamma}} (\beta\gamma + C|\Delta_k|^2), \quad (22)$$

where

$$\bar{H}_t^\gamma = \int_0^t g_\gamma(\underline{s}) \langle \nabla V(\bar{X}_{\underline{s}}), \sigma(\bar{X}_{\underline{s}}) dB_s^H \rangle = \sum_{i,j} \int_0^t g_\gamma(\underline{s}) (\nabla V)_i(\bar{X}_{\underline{s}}), \sigma_{i,j}(\bar{X}_{\underline{s}}) d(B_s^H)^j.$$

with $g_\gamma(s) = (1 - \frac{\alpha\gamma}{2})^{-\frac{s}{\gamma}}$. For every $(i, j) \in \{1, \dots, d\} \times \{1, \dots, q\}$, set $f_\gamma^{i,j}(s, x) = g_\gamma(s) (\nabla V)_i(x) \sigma_{i,j}(x)$. Using that $\sup_{t \in [0, T], \gamma \in (0, \gamma_0]} |g'_\gamma(t)| < +\infty$, we check that $(g_\gamma(\cdot))_{\gamma \in (0, \gamma_0]}$ is a family of Lipschitz continuous functions such that $\sup_{\gamma \in (0, \gamma_0]} [g_\gamma]_{\text{Lip}} < +\infty$. Furthermore, $(\nabla V)_i$ and $\sigma_{i,j}$ being respectively Lipschitz continuous and bounded Lipschitz continuous functions, we deduce that $(f_\gamma^{i,j})_{\gamma \in (0, \gamma_0]}$ satisfies (14) with $r = 1$. Applying Lemma 2, we obtain that for every $\theta \in (\frac{1}{2}, H)$,

$$\frac{|\bar{H}_{\underline{t}}^\gamma - \bar{H}_{\underline{s}}^\gamma|}{(\underline{t} - \underline{s})^\theta} \leq C_T \left[(1 + |\bar{X}_{\underline{s}}^\gamma|) + C_T(\underline{t} - \underline{s})^\theta (1 + |\bar{X}_{\underline{s}}^\gamma|^2 + (\|\bar{Z}^\gamma\|_{\theta, \gamma}^{\underline{s}, \underline{t}-\gamma})^2) \right] \|B^H\|_{\theta, T}.$$

Now, if $\underline{t} - \underline{s} \leq \eta(\omega)$ defined by (20),

$$\|\bar{Z}^\gamma\|_{\theta, \gamma}^{\underline{s}, \underline{t}-\gamma} \leq \left(\|\sigma\|_\infty + C_T(1 + |\bar{X}_s|) \eta^\theta \right) \|B^H\|_{\theta, T}$$

but owing to the definition of η , we have *a.s.*

$$\|B^H(\omega)\|_{\theta, T} \eta^\theta \leq C_T$$

where C_T is a deterministic positive number so that

$$(\|\bar{Z}^\gamma\|_{\theta, \gamma}^{\underline{s}, \underline{t}-\gamma})^2 \leq C_T(\|B^H\|_{\theta, T}^2 + 1 + |\bar{X}_s|^2).$$

Thus,

$$|\bar{H}_{\underline{t}}^\gamma - \bar{H}_{\underline{s}}^\gamma| \leq C_T \left[(1 + |\bar{X}_{\underline{s}}^\gamma|)(\underline{t} - \underline{s})^\theta + (\underline{t} - \underline{s})^{2\theta} (1 + |\bar{X}_{\underline{s}}^\gamma|^2 + \|B^H\|_{\theta, T}^2) \right] \|B^H\|_{\theta, T}.$$

Using that $|ab| \leq 2^{-1}(|a|^2 + |b|^2)$ and that $1 + |x| \leq C\sqrt{V}(x)$, we have

$$(1 + |\bar{X}_{\underline{s}}^\gamma|)(\underline{t} - \underline{s})^\theta \|B^H\|_{\theta, T} \leq C(V(\bar{X}_{\underline{s}})(\underline{t} - \underline{s})^{2\theta} + \|B^H\|_{\theta, T}^2).$$

It follows that there exists $C_T > 0$ such that for every $\varepsilon > 0$

$$|\bar{H}_{\underline{t}}^\gamma - \bar{H}_{\underline{s}}^\gamma| \leq \varepsilon(\underline{t} - \underline{s}) V(\bar{X}_{\underline{s}}^\gamma) \left(\frac{C_T(\underline{t} - \underline{s})^{2\theta-1} (1 + \|B^H\|_{\theta, T})}{\varepsilon} \right) + C_T(\|B^H\|_{\theta, T}^2 + (\underline{t} - \underline{s})^{2\theta} \|B^H\|_{\theta, T}^3).$$

Now, we choose $\tilde{\eta}_\varepsilon \in (0, \eta)$

$$\frac{C_T(\tilde{\eta}_\varepsilon)^{2\theta-1} (1 + \|B^H\|_{\theta, T})}{\varepsilon} \leq 1.$$

More precisely, we set $\tilde{\eta}_\varepsilon = [(C_T(1 + \|B^H\|_{\theta,T})^{-1}\varepsilon)^{\frac{1}{2\theta-1}} \wedge \eta]$. Thus, we obtain that for every $0 \leq s \leq t \leq T$ such that $\underline{t} - \underline{s} \leq \tilde{\eta}_\varepsilon$,

$$|\bar{H}_{\underline{t}}^\gamma - \bar{H}_{\underline{s}}^\gamma| \leq \varepsilon(\underline{t} - \underline{s})V(\bar{X}_{\underline{s}}^\gamma) + C_T(\|B^H\|_{\theta,T}^2 + (\underline{t} - \underline{s})^{2\theta}\|B^H\|_{\theta,T}^3). \quad (23)$$

Then, we can set $\varepsilon_0 > 0$ in order that there exists $\delta > 0$ such that

$$\forall x \in [0, 1], \quad e^{-\frac{\alpha}{2}x}(1 + \varepsilon_0 x) \leq 1 - \delta x.$$

Thus, setting $\tilde{\eta} := \tilde{\eta}_{\varepsilon_0}$ and plugging the two previous controls in (22), it follows that for every $k \in \{1, \dots, \lfloor \frac{T}{\tilde{\eta}} \rfloor\}$,

$$V(\bar{X}_{\underline{k}\tilde{\eta}}) \leq V(\bar{X}_{(\underline{k-1})\tilde{\eta}})(1 - \delta\tilde{\eta}) + C_T(1 + \|B^H\|_{\theta,T}^3) + \sum_{l=\frac{(k-1)\tilde{\eta}}{\gamma}+1}^{\frac{k\tilde{\eta}}{\gamma}} (\beta\gamma + C|\Delta_l|^2).$$

An iteration yields for every $k \in \{1, \dots, \lfloor \frac{T}{\tilde{\eta}} \rfloor\}$:

$$\begin{aligned} V(\bar{X}_{\underline{k}\tilde{\eta}}) &\leq V(x)(1 - \delta\tilde{\eta})^k + C_T\|B^H\|_{\theta,T}^3 \sum_{m=1}^k (1 - \delta\tilde{\eta})^{k-m} \\ &\quad + \sum_{m=1}^k (1 - \delta\tilde{\eta})^{k-m} \sum_{l=\frac{(m-1)\tilde{\eta}}{\gamma}+1}^{\frac{m\tilde{\eta}}{\gamma}} (\beta\gamma + C|\Delta_l|^2). \end{aligned}$$

Now, on the one hand, $\sum_{l=1}^k (1 - \delta\tilde{\eta})^{k-l} \leq \tilde{\eta}^{-1}$ and owing to the definition of $\tilde{\eta}$ (and of η), we have $\tilde{\eta}^{-1} \leq C(1 + \|B^H\|_{\theta,T}^{\frac{1}{2\theta-1}})$. It follows that there exists a function P_θ with polynomial growth such that

$$C_T\|B^H\|_{\theta,T}^3 \sum_{m=1}^k (1 - \delta\tilde{\eta})^{k-m} \leq P(\|B^H\|_{\theta,T}).$$

In the sequel of the proofs the index θ in P_θ , which recall the dependance of the polynomial in θ is dropped. On the other hand, since $(1 - \delta\tilde{\eta})^{k-m} \leq 1$, we also have

$$\sum_{m=1}^k (1 - \delta\tilde{\eta})^{k-m} \sum_{l=\frac{(m-1)\tilde{\eta}}{\gamma}+1}^{\frac{m\tilde{\eta}}{\gamma}} (\beta\gamma + C|\Delta_l|^2) \leq \sum_{u=1}^{\lfloor \frac{k\tilde{\eta}}{\gamma} \rfloor} (\beta\gamma + C|\Delta_u|^2) \leq \beta k\tilde{\eta} + C \sum_{u=1}^{\lfloor \frac{k\tilde{\eta}}{\gamma} \rfloor} |\Delta_u|^2.$$

We deduce that for every $k \in \{1, \dots, \lfloor \frac{T}{\tilde{\eta}} \rfloor\}$:

$$V(\bar{X}_{\underline{k}\tilde{\eta}}) \leq V(x)(1 - \delta\tilde{\eta})^k + P(\|B^H\|_{\theta,T}) + CQ_\gamma(B_t^H, 0 \leq t \leq T), \quad (24)$$

where P is a function with polynomial growth and Q_γ is defined in the lemma. Applying this inequality with $k = \lfloor \frac{T}{\tilde{\eta}} \rfloor$, we obtain

$$V(\bar{X}_{\underline{T}}) \leq V(x)(1 - \delta\tilde{\eta})^{\lfloor \frac{T}{\tilde{\eta}} \rfloor} + P(\|B^H\|_{\theta,T}) + CQ_\gamma(B_t^H, 0 \leq t \leq T). \quad (25)$$

Finally, we want to control $V(\bar{X}_T) - V(\bar{X}_{\underline{T}})$. The function ∇V being sublinear and D^2V being bounded, we deduce from the Taylor formula that for every $x, y \in \mathbb{R}^d$,

$$V(y) \leq V(x) + C(|x| \cdot |y - x| + |y - x|^2).$$

Applying this inequality with $x = \bar{X}_{\underline{T}}$ and $y = \bar{X}_T$ and taking advantage of the assumptions on b , we have

$$V(\bar{X}_T) \leq V(\bar{X}_{\underline{T}}) + C [\gamma(1 + |\bar{X}_{\underline{T}}|^2) + (1 + |\bar{X}_{\underline{T}}|)|B_T^H - B_{\underline{T}}^H| + |B_T^H - B_{\underline{T}}^H|^2] \quad (26)$$

$$\leq V(\bar{X}_{\underline{T}})(1 + C\gamma) + C(1 + \|B^H\|_{\theta,T}^2), \quad (27)$$

where in the second line, we again used the elementary inequality $|ab| \leq 2^{-1}(|a|^2 + |b|^2)$ and the fact that $|x|^2 \leq CV(x)$. Combined with (25), the previous inequality yields the previous inequality yields:

$$V(\bar{X}_T) \leq V(x)(1 - \delta\tilde{\eta})^{\lfloor \frac{T}{\tilde{\eta}} \rfloor} (1 + C\gamma) + P(\|B^H\|_{\theta,T}) + CQ_\gamma(B_t^H, 0 \leq t \leq T),$$

where P again denotes a function with polynomial growth. Finally, since $(1 - \delta\tilde{\eta})^{\lfloor \frac{T}{\tilde{\eta}} \rfloor} \leq e^{-\delta T}$ there exists $\gamma_0 > 0$ and $\rho \in (0, 1)$ such that for every $\gamma \in (0, \gamma_0)$,

$$(1 - \delta\tilde{\eta})^{\lfloor \frac{T}{\tilde{\eta}} \rfloor} (1 + C\gamma) \leq \rho.$$

Inequality (10) follows.

Case $p > 1$: For every $u, v \in \mathbb{R}$, $|u+v|^p = |u|^p + p|u+\kappa v|^{p-1} \text{sgn}(u+\kappa v)v$ with $\kappa \in [0, 1]$. Since $|u+\kappa v|^{p-1} \leq 2^{p-1}(|u|^{p-1} + |v|^{p-1})$, we deduce that $|u+v|^p \leq |u|^p + c_p(|v| \cdot |u|^{p-1} + |v|^p)$ for every $u, v \in \mathbb{R}$ and $p \geq 1$. Then, by the Young inequality, it follows that for every $\varepsilon > 0$, there exists $c_{\varepsilon,p} > 0$ such that $|u+v|^p \leq (1+\varepsilon)|u|^p + c_{\varepsilon,p}|v|^p$ for every $u, v \in \mathbb{R}$ and $p \geq 1$. Applying this inequality, we deduce from the case $p = 1$ that

$$V^p(\bar{X}_1^\gamma) \leq \rho^p(1 + \varepsilon)V^p(x) + C(P(\|B^H\|_{\theta,1}) + CQ_\gamma(B_t^H, 0 \leq t \leq 1))^p.$$

Since $\rho < 1$, we can choose $\varepsilon > 0$ such that $\tilde{\rho} = \rho^p(1 + \varepsilon) < 1$. Using again the elementary inequality $|u+v|^p \leq 2^{p-1}(|u|^p + |v|^p)$, we obtain

$$V^p(\bar{X}_1^\gamma) \leq \tilde{\rho}V^p(x) + P_{p,\theta}(\|B^H\|_{\theta,1}) + CQ_\gamma^p(B_t^H, 0 \leq t \leq 1)$$

where $P_{p,\theta}$ is a polynomial function.

Now, let us focus on (12). We only give the main ideas of the proof when $p = 1$ (the extension to $p > 1$ again follows from the inequality $|u+v|^p \leq 2^{p-1}(|u|^p + |v|^p)$). By (24), the announced inequality holds taking the supremum of the left-hand side of (12) for every $k\tilde{\eta}$ with $k \in \{1, \dots, \lfloor \frac{T}{\tilde{\eta}} \rfloor\}$. Then, for every $t \in [(k-1)\tilde{\eta}, k\tilde{\eta}]$, it remains to control (uniformly in k) $V(\bar{X}_t)$ in terms of $V(\bar{X}_{(k-1)\tilde{\eta}})$. By (21) and (23), we obtain such a control for every discretization time between $(k-1)\tilde{\eta}$ and $k\tilde{\eta}$. Then, it is enough to control uniformly $V(\bar{X}_t)$ in terms of $V(\bar{X}_{\underline{T}})$. This can be done similarly as in inequality (26).

(ii) Let $s, t \in [0, T]$ with $0 \leq s < t \leq T$. We have

$$\bar{X}_t^\gamma - \bar{X}_s^\gamma = \int_s^t b(\bar{X}_u^\gamma) du + \bar{Z}_t^\gamma - \bar{Z}_s^\gamma.$$

First, since $|b(x)| \leq C\sqrt{V}(x)$,

$$|\int_s^t b(\bar{X}_u^\gamma) du| \leq (t-s) \sup_{u \in [0, T]} \sqrt{V}(\bar{X}_u).$$

Second, we focus on the increment of \bar{Z}^γ . By Lemma 3, for every $u, v \in [0, T]$ such that $\underline{v} - \underline{u} \leq \eta$ (where η is given by (20)),

$$|\bar{Z}_v^\gamma - \bar{Z}_u^\gamma| \leq (\underline{v} - \underline{u})^\theta \left(\|\sigma\|_\infty + C_T(1 + \sup_{s \in [0, T]} |\bar{X}_s|) \eta^\theta \right) \|B^H\|_{\theta, T}.$$

Using the concavity of $x \mapsto x^\theta$ on \mathbb{R}_+ , we have for every $s_1, s_2 \in [0, T]$ being such that $|s_2 - s_1| \leq \gamma$,

$$|\bar{Z}_{s_2}^\gamma - \bar{Z}_{s_1}^\gamma| \leq 2^{1-\theta} \|\sigma\|_\infty (s_2 - s_1)^\theta \|B^H\|_{\theta, T}$$

and we derive that for every $u, v \in [0, T]$ with $|u - v| \leq \eta$,

$$|\bar{Z}_v^\gamma - \bar{Z}_u^\gamma| \leq C_T(v - u)^\theta \left(\|\sigma\|_\infty + (1 + \sup_{s \in [0, T]} |\bar{X}_s|) \eta^\theta \right) \|B^H\|_{\theta, T}.$$

Now, by the very definition of η , we have $\eta^\theta \|B^H\|_{\theta, T} \leq 1$. Then, since $|x| \leq CV(x)$, we deduce from the first part of this proposition that for every $u, v \in [0, T]$ with $|u - v| \leq \eta$:

$$|\bar{Z}_v^\gamma - \bar{Z}_u^\gamma| \leq C_T(v - u)^\theta (V(x) + \tilde{P}(\|B^H\|_{\theta, T}) + CQ_\gamma(B_t^H, 0 \leq t \leq T)). \quad (28)$$

where \tilde{P} is a polynomial function and Q_γ is defined in Lemma 3.

We want now to make use of the preceding inequality to control $\bar{Z}_t^\gamma - \bar{Z}_s^\gamma$ for every $0 \leq s < t \leq T$. We divide $[s, t]$ in intervals of length lower than η . More precisely, setting $s_k = s + k\lfloor \eta \rfloor$, we have

$$\bar{Z}_t^\gamma - \bar{Z}_s^\gamma = \bar{Z}_t^\gamma - \bar{Z}_{s_{\lfloor \frac{t-s}{\eta} \rfloor}}^\gamma + \sum_{i=1}^{\lfloor \frac{t-s}{\eta} \rfloor} \bar{Z}_{s_k}^\gamma - \bar{Z}_{s_{k-1}}^\gamma.$$

Then, we deduce from (28) that

$$\begin{aligned} |\bar{Z}_t^\gamma - \bar{Z}_s^\gamma| &\leq C_T \left((t - s_{\lfloor \frac{t-s}{\eta} \rfloor})^\theta + \lfloor \frac{t-s}{\eta} \rfloor \eta^\theta \right) (V(x) + \tilde{P}(\|B^H\|_{\theta, T}) + CQ_\gamma(B_t^H, 0 \leq t \leq T)) \\ &\leq C_T \left((t - s)^\theta + (t - s) \eta^{\theta-1} \right) (V(x) + \tilde{P}(\|B^H\|_{\theta, T}) + CQ_\gamma(B_t^H, 0 \leq t \leq T)). \end{aligned}$$

Thus, using (28) if $t - s \leq \eta$ or the fact that $(t - s) \eta^{\theta-1} \leq (t - s)^\theta$ if $t - s \geq \eta$, we deduce that there exists $C_T > 0$ such that for every $0 \leq s < t \leq T$,

$$|\bar{Z}_t^\gamma - \bar{Z}_s^\gamma| \leq C_T(t - s)^\theta (V(x) + \tilde{P}(\|B^H\|_{\theta, T}) + CQ_\gamma(B_t^H, 0 \leq t \leq T)).$$

The result follows.

4 Tightness properties

In the following proposition, we obtain some *a.s.* tightness results for the sequence $(\mathcal{P}^{(n, \gamma)}(\omega, d\alpha))_{n \geq 1}$.

Using that the controls established in Proposition 3 are uniform in γ , we also show that tightness properties also hold for the set of its limiting measures $(\mathcal{U}^{(\infty, \gamma)}(\omega, \theta))_\gamma$ defined by

$$\mathcal{U}^{(\infty, \gamma)}(\omega, \theta) = \left\{ \mu \in \bar{\mathcal{C}}^\theta(\mathbb{R}_+, \mathbb{R}^d), \exists (n_k(\omega))_{k \geq 1}, \mathcal{P}^{(n_k(\omega), \gamma)}(\omega, d\alpha) \xrightarrow{n \rightarrow +\infty} \mu \right\}.$$

PROPOSITION 4. Assume **(C)**. Then, there exists $\gamma_0 > 0$ such that,

(i) For every $\gamma \in (0, \gamma_0]$ and $p \geq 1$, a.s.,

$$\limsup_{n \rightarrow +\infty} \frac{1}{n} \sum_{k=0}^n V^p(\bar{X}_{\gamma(k-1)}^\gamma) \leq C_p \mathbb{E}[|P_p(\|B^H\|_{\theta,1})| + Q_\gamma^p(B_t^H, 0 \leq t \leq 1)] < +\infty.$$

where C does not depend on γ and P_p is a polynomial function and Q_γ is defined by (11).

(ii) For every $\theta \in (1/2, H)$, for every $\gamma \in (0, \gamma_0]$, $(\mathcal{P}^{(n,\gamma)}(\omega, d\alpha))_{n \geq 1}$ is almost surely tight on $\bar{\mathcal{C}}^\theta(\mathbb{R}_+, \mathbb{R}^d)$.

(iii) For every $\theta \in (1/2, H)$, $(\mathcal{U}^{(\infty,\gamma)}(\omega, \theta))_{\gamma \in (0, \gamma_0]}$ is a.s. tight in $\bar{\mathcal{C}}^\theta(\mathbb{R}_+, \mathbb{R}^d)$.

Proof. (i) **Case** $p = 1$: We first focus on the sequence $(\frac{1}{N} \sum_{k=0}^{N-1} V(\bar{X}_k^\gamma))_{N \geq 1}$. We set

$$\forall k \geq 0, \quad (\delta_k B^H)_t = B_{k\gamma+t}^H - B_{k\gamma}^H.$$

By Proposition 3 applied with $T = 1$, we have for every $k \geq 1$

$$V(\bar{X}_k^\gamma) \leq \rho V(\bar{X}_{k-1}^\gamma) + P(\|\delta_{(k-1)} B^H\|_{\theta,1}) + CQ_\gamma((\delta_{(k-1)} B^H)_t, 0 \leq t \leq 1)$$

with $\rho \in (0, 1)$. An iteration yields for every $k \geq 1$

$$V(\bar{X}_k^\gamma) \leq \rho^k V(x) + \sum_{l=0}^{k-1} \rho^{k-1-l} P(\|\delta_l B^H\|_{\theta,1}) + CQ_\gamma((\delta_l B^H)_t, 0 \leq t \leq 1).$$

Setting $U_l = P(\|\delta_l B^H\|_{\theta,1}) + CQ_\gamma((\delta_l B^H)_t, 0 \leq t \leq 1)$ and summing over k , we obtain

$$\begin{aligned} \frac{1}{N} \sum_{k=0}^{N-1} V(\bar{X}_k^\gamma) &\leq \frac{V(x)}{N(1-\rho)} + \frac{1}{N} \sum_{k=0}^{N-1} \sum_{l=0}^{k-1} \rho^{k-1-l} U_l \\ &\leq \frac{V(x)}{N(1-\rho)} + \frac{1}{N} \sum_{l=0}^{N-2} U_l \sum_{k=l+1}^N \rho^{k-1-l} \leq \frac{V(x)}{N(1-\rho)} + \frac{1}{N(1-\rho)} \sum_{l=0}^{N-2} U_l. \end{aligned}$$

Let us remark that since B^H is a $\bar{\mathcal{C}}^\theta([0, 1], \mathbb{R}^q)$ valued Gaussian random variable, the norm $\|B^H\|_{\theta,1}$ has finite moments of every order, which is classical consequence of Fernique Lemma. Moreover

$$\mathbb{E}[Q_\gamma(B_t^H, 0 \leq t \leq 1)] \leq C\gamma^{2H-1}.$$

Hence

$$h(\gamma) := \mathbb{E}[P(\|B^H\|_{\theta,1}) + CQ_\gamma(B_t^H, 0 \leq t \leq 1)] < +\infty. \quad (29)$$

Since $(\delta_l B^H)$ is ergodic and that $h(\gamma) < +\infty$, We have

$$\frac{1}{N} \sum_{l=0}^{N-2} U_l \xrightarrow{n \rightarrow +\infty} \mathbb{E}[P(\|B^H\|_{\theta,1}) + CQ_\gamma(B_t^H, 0 \leq t \leq 1)] \quad a.s. \quad (30)$$

and it follows that

$$\limsup_{N \rightarrow +\infty} \frac{1}{N} \sum_{k=0}^{N-1} V(\bar{X}_k^\gamma) \leq \frac{1}{1-\rho} \mathbb{E}[P(\|B^H\|_{\theta,1}) + CQ_\gamma(B_t^H, 0 \leq t \leq 1)] \quad a.s. \quad (31)$$

We want now to use this result to control the a.s. asymptotic behavior of $(\frac{1}{n} \sum_{k=0}^{n-1} V(\bar{X}_{\gamma k}^\gamma))_{n \geq 1}$. By the second point of Proposition 3(i), for every $k \geq 0$,

$$\sup_{l \in [\lfloor \frac{k}{\gamma} \rfloor + 1, \lfloor \frac{k+1}{\gamma} \rfloor]} V(\bar{X}_l^\gamma) \leq C (V(\bar{X}_k^\gamma) + P(\|\delta_k B^H\|_{\theta,1}) + CQ_\gamma((\delta_k B^H)_t, 0 \leq t \leq 1)).$$

As a consequence, setting $N = \lfloor \gamma(n-1) \rfloor + 1$, we have

$$\begin{aligned} \frac{1}{n} \sum_{l=0}^{n-1} V(\bar{X}_{\gamma l}^\gamma) &\leq \frac{N}{n} \frac{1}{N} \left(V(x) + \sum_{k=0}^{N-1} \sum_{\lfloor \frac{k}{\gamma} \rfloor + 1}^{\lfloor \frac{k+1}{\gamma} \rfloor} V(\bar{X}_{\gamma l}^\gamma) \right) \\ &\leq C(\gamma + \frac{1}{n})(\frac{1}{\gamma} + 1) \left(\frac{1}{N} \sum_{k=0}^{N-1} (V(\bar{X}_k^\gamma) + P(\|\delta_k B^H\|_{\theta,1}) + CQ_\gamma((\delta_k B^H)_t, 0 \leq t \leq 1)) \right). \end{aligned}$$

Using (30) and (31), the result follows when $p = 1$.

The proof when $p > 1$ is very similar to the case $p = 1$ and is left to the reader.

(ii) By (i), $(\mathcal{P}_0^{(n,\gamma)}(\omega, dx))$ is tight on \mathbb{R}^d (since V is coercive). Owing to some classical tightness results in Hölder spaces (see *e.g.* [27], Theorem 1.4), we deduce that we have only to prove that for every $T > 0$, for every $\theta \in (1/2, H)$, for every $\varepsilon > 0$,

$$\limsup_{\delta \rightarrow 0} \limsup_{n \rightarrow +\infty} \frac{1}{n} \sum_{k=1}^n \mathbf{1}_{\{\omega_{\theta,T}(\bar{X}_{\gamma(k-1)+.}^\gamma, \delta) \geq \varepsilon\}} = 0,$$

where we recall that

$$\forall T > 0, \quad \omega_{\theta,T}(f, \delta) := \sup_{0 \leq s < t < T, 0 \leq |t-s| \leq \delta} \frac{|f(t) - f(s)|}{|t-s|^\theta}.$$

By Proposition 3 (ii) with $\theta' \in (\theta, H)$,

$$\sup_{0 \leq s < t \leq T} \frac{|\bar{X}_t^\gamma - \bar{X}_s^\gamma|}{(t-s)^\theta} \leq C_T (t-s)^{\theta'-\theta} (V(x) + \tilde{P}(\|B^H\|_{\theta,T}) + Q_\gamma(B_t^H, 0 \leq t \leq T))$$

so that for every $s, t \in [0, T]$ such that $s < t$ and $t-s \leq \delta$,

$$\sup_{0 \leq s < t \leq T} \frac{|\bar{X}_t^\gamma - \bar{X}_s^\gamma|}{(t-s)^\theta} \leq C_T \delta^{\theta'-\theta} (V(x) + \tilde{P}(\|B^H\|_{\theta,T}) + Q_\gamma(B_t^H, 0 \leq t \leq T)).$$

As in (i), this property can be extended to the shifted process: we have for every $k \geq 0$

$$\sup_{0 \leq s < t \leq T} \frac{|\bar{X}_{\gamma k+t}^\gamma - \bar{X}_{\gamma k+s}^\gamma|}{(t-s)^\theta} \leq C_T \delta^{\theta'-\theta} (V(\bar{X}_{\gamma k}^\gamma) + \tilde{P}(\|\delta_k B^H\|_{\theta,T}) + Q_\gamma((\delta_k B^H)_t, 0 \leq t \leq T))$$

By (i) and the ergodic properties of the increments of B^H (see (30) for similar arguments), we deduce that

$$\limsup_{n \rightarrow +\infty} \frac{1}{n} \sum_{k=1}^n \mathbf{1}_{\omega_{\theta,T}(\bar{X}_{\gamma(k-1)+.}^\gamma, \delta)} \leq C \delta^{\theta'-\theta}. \quad (32)$$

The result follows.

(iii) Let $\theta \in (1/2, H)$ and denote by $\mu^{(\gamma)}$ an element of $\mathcal{U}^{(\infty,\gamma)}(\omega, \theta)$ and by $\mu_t^{(\gamma)}$ its marginals. By (29) and (31),

$$\forall \gamma \in (0, \gamma_0], \quad \mu_0^{(\gamma)}(V) \leq \frac{1}{1-\rho} h(\gamma)$$

where ρ does not depend on γ . Since $H > 1/2$, $\sup_{\gamma \in (0, \gamma_0]} h(\gamma) < +\infty$. It follows that $\mathcal{U}_0^{(\infty, \gamma)}(\omega, \theta)$ is *a.s.* tight in \mathbb{R}^d (where $\mathcal{U}_0^{(\infty, \gamma)}(\omega, \theta)$ stands for the set of initial distributions $\mu_0^{(\gamma)}$).

Now, since C does not depend on γ in (32), we also have for every $T > 0$, $\delta > 0$ and $\varepsilon > 0$ for every $\theta' > \theta$:

$$\forall \gamma \in (0, \gamma_0], \quad \mu^{(\gamma)}(\mathbf{1}_{\{\omega_{\theta, T}(\cdot, \delta) \geq \varepsilon\}}) \leq C\delta^{\theta' - \theta}.$$

and the announced result follows again from Theorem 1.4 of [27]. \square

5 Identification of the weak limits

5.1 Weak limits of $(\mathcal{P}^{(n, \gamma)}(\omega, d\alpha))_{n \geq 1}$

We have the following result:

PROPOSITION 5. *Assume (C) and let $\mathcal{P}^{(\infty, \gamma)}(\omega, d\alpha)$ denote a weak limit of $((\mathcal{P}^{(n, \gamma)}(\omega, d\alpha))_{n \geq 1})$. Then, $\mathcal{P}^{(\infty, \gamma)}(\omega, d\alpha)$ is *a.s.* an adapted stationary solution of (6).*

REMARK 4. In the following proof, we will prove some properties “for every function f , for almost every ω ” and conclude that “for almost every ω , for every function f ” the property is true. For the sake of completeness, we recall here that such inversions are rigorous since we work on Polish spaces (in which the distributions and the weak convergence are characterized by some countable family of bounded continuous functions).

Proof. In the proof, we denote by $(\tilde{\mathcal{P}}^{(n)}(\omega, d\alpha, d\beta))_{n \geq 1}$, the sequence of probability measures on $\bar{\mathcal{C}}^\theta(\mathbb{R}_+, \mathbb{R}^d) \times \bar{\mathcal{C}}^\theta(\mathbb{R}, \mathbb{R}^q)$ with $\frac{1}{2} < \theta < H$ defined by

$$\tilde{\mathcal{P}}^{(n, \gamma)}(\omega, d\alpha, d\beta) = \frac{1}{n} \sum_{k=1}^n \delta_{(\bar{X}_{\gamma(k-1)+\cdot}^\gamma(\omega), B_{(k-1)\gamma+}^H(\omega) - B_{(k-1)\gamma}^H(\omega))}(d\alpha, d\beta)$$

where $(B_t^H)_{t \in \mathbb{R}}$ is the fractional Brownian motion used to build the Euler scheme (9). First, let us recall that by Proposition 4 (ii), $(\mathcal{P}^{(n, \gamma)}(\omega, d\alpha))_{n \geq 1}$ is *a.s.* tight. Thus, we can consider a weak limit $\mathcal{P}^{(\infty, \gamma)}(\omega, d\alpha)$. Second, one checks that $(\tilde{\mathcal{P}}^{(n, \gamma)}(\omega, d\alpha, d\beta))_{n \geq 1}$ is also almost surely tight since each of its margins have this property. Indeed, for the first margin, it is again (ii) of Proposition 4. For the second margin, we use that $(B_t^H)_{t \in \mathbb{R}}$ is ergodic under the transformation $T_\gamma : \bar{\mathcal{C}}^\theta(\mathbb{R}, \mathbb{R}^d) \rightarrow \bar{\mathcal{C}}^\theta(\mathbb{R}, \mathbb{R}^q)$ defined by $(T_\gamma(\omega))_t = \omega(\gamma + t) - \omega(\gamma)$ (see *e.g.* [22]). In particular,

$$\frac{1}{n} \sum_{k=1}^n \delta_{B_{(k-1)\gamma+}^H - B_{(k-1)\gamma}^H}(d\beta) \tag{33}$$

is converging almost surely to the distribution of $(B_t^H)_{t \in \mathbb{R}}$ (on $\bar{\mathcal{C}}^\theta(\mathbb{R}, \mathbb{R}^d)$). Hence, the sequence $(\tilde{\mathcal{P}}^{(n)}(\omega, d\alpha, d\beta))_{n \geq 1}$ is almost surely tight (and thus relatively compact). Then, if $\mathcal{P}^{(\infty, \gamma)}(\omega, d\alpha)$ is the limit of a subsequence of $(\mathcal{P}^{(n, \gamma)}(\omega, d\alpha))_{n \geq 1}$, maybe with the help of a second extraction, it follows that *a.s.*, there exists a subsequence $(n_k(\omega))_{k \geq 0}$ such that

$$\mathcal{P}^{(n_k, \gamma)}(\omega, d\alpha) \xrightarrow{k \rightarrow +\infty} \mathcal{P}^{(\infty, \gamma)}(\omega, d\alpha) \quad \text{and} \quad \tilde{\mathcal{P}}^{(n_k, \gamma)}(\omega, d\alpha, d\beta) \xrightarrow{n_k \rightarrow +\infty} \tilde{\mathcal{P}}^{(\infty, \gamma)}(\omega, d\alpha, d\beta) \tag{34}$$

where the first margin of $\tilde{\mathcal{P}}^{(\infty, \gamma)}(\omega, d\alpha, d\beta)$ is obviously $\mathcal{P}^{(\infty, \gamma)}(\omega, d\alpha)$ and the second one is *a.s.* the distribution of $(B_t^H)_{t \in \mathbb{R}}$ (thanks to (33)). Let us also denote by $(X_t^{(\infty, \gamma)}, B_t^H)$

the coordinate process on $\bar{C}^\theta(\mathbb{R}_+, \mathbb{R}^d) \times \bar{C}^\theta(\mathbb{R}, \mathbb{R}^q)$ endowed with the probability $\tilde{\mathcal{P}}^{(\infty, \gamma)}$. For $(\alpha, \beta) \in \bar{C}^\theta(\mathbb{R}_+, \mathbb{R}^d) \times \bar{C}^\theta(\mathbb{R}, \mathbb{R}^q)$ we consider the following function

$$\tilde{\Phi}^\gamma(\alpha, \beta)_t := \alpha_0 + \int_0^t b(\tilde{\Phi}^\gamma(\alpha, \beta)_{\underline{s}_\gamma}) ds + \int_0^t \sigma(\tilde{\Phi}^\gamma(\alpha, \beta)_{\underline{s}_\gamma}) d\beta_s. \quad (35)$$

Please remark that $\tilde{\Phi}^\gamma$ is slightly different from Φ^γ in the way it handles the initial condition but

$$\tilde{\Phi}^\gamma(\alpha, \beta) = \Phi^\gamma(a, \beta)$$

for every α such that $\alpha_0 = a$. For $t, K > 0$ let us denote by $F_{t,K}$ the function defined on $\bar{C}^\theta(\mathbb{R}_+, \mathbb{R}^d) \times \bar{C}^\theta(\mathbb{R}, \mathbb{R}^q)$ by $F_{t,K}(\alpha, \beta) = \sup_{0 \leq s \leq t} |\alpha_s - \tilde{\Phi}^\gamma(\alpha, \beta)_s| \wedge K$ where $\beta_+ = (\beta(t))_{t \geq 0}$. The function $F_{t,K}$ is bounded continuous on $\bar{C}^\theta(\mathbb{R}_+, \mathbb{R}^d) \times \bar{C}^\theta(\mathbb{R}, \mathbb{R}^q)$.

Then,

$$\mathbb{E}(F_{t,K}(X^{(\infty, \gamma)}, B^H)) = \lim_{n_l \rightarrow \infty} \frac{1}{n_l} \sum_{k=1}^{n_l} F_{t,K}(\bar{X}_{(k-1)\gamma+}^\gamma, B_{(k-1)\gamma+}^H - B_{(k-1)\gamma}^H).$$

By definition of the Euler scheme (even though it is shifted), we have for every $k \geq 1$, $F_{t,K}(\bar{X}_{(k-1)\gamma+}^\gamma, B_{(k-1)\gamma+}^H - B_{(k-1)\gamma}^H) = 0$ almost surely, and

$$X^{(\infty, \gamma)} = \tilde{\Phi}^\gamma(X^{(\infty, \gamma)}, B^H)$$

almost surely, which ensures that the pair $(X^{(\infty, \gamma)}, B^H)$ is a solution of (6).

The stationarity of $X^{(\infty, \gamma)}$ follows from the construction. Actually, using the convergence of $(\mathcal{P}^{(n, \gamma)}(\omega, d\alpha))$, we have for every bounded continuous functional $F : \bar{C}^\theta(\mathbb{R}_+, \mathbb{R}^d) \rightarrow \mathbb{R}$,

$$\frac{1}{n} \sum_{k=1}^n F(\bar{X}_{\gamma(k-1)+t+}^\gamma) - F(\bar{X}_{\gamma(k-1)+}^\gamma) \xrightarrow{n \rightarrow +\infty} \mathbb{E}[F(X_{t+}^{(\infty, \gamma)})] - \mathbb{E}[F(X_{\cdot}^{(\infty, \gamma)})]$$

and owing to a change of variable, it is obvious that for every $t \in \gamma\mathbb{N}$,

$$\frac{1}{n} \sum_{k=1}^n F(\bar{X}_{\gamma(k-1)+t+}^\gamma) - F(\bar{X}_{\gamma(k-1)+}^\gamma) \xrightarrow{n \rightarrow +\infty} 0$$

It follows that for every $t \in \gamma\mathbb{N}$, for every F ,

$$\mathbb{E}[F(X_{t+}^{(\infty, \gamma)})] = \mathbb{E}[F(X_{\cdot}^{(\infty, \gamma)})].$$

This property implies that $X^{(\infty, \gamma)}$ is stationary.

We now focus on the adaptation of $X^{(\infty, \gamma)}$. In this step, we need to introduce, for a subset D of \mathbb{R} that contains 0, the Polish space $\mathcal{W}_{\theta, \delta}(D)$ that denotes the completion of $\mathcal{C}_0^\infty(D, \mathbb{R}^q)$ (space of \mathcal{C}^∞ -functions $f : D \rightarrow \mathbb{R}^q$ with compact support and $f(0) = 0$) for the norm

$$\|f\| = \sup_{s, t \in D} \frac{|f(t) - f(s)|}{|t - s|^\theta (1 + |t|^\delta + |s|^\delta)}.$$

This space is convenient to obtain some Feller properties for the conditional distribution of the fractional Brownian motion given its past. More precisely, by Lemmas 4.1 to 4.3 of [14], the paths of B^H belong *a.s.* to $\mathcal{W}_{\theta, \delta}(\mathbb{R})$ when $\theta \in (1/2, H)$ and $\theta + \delta \in (H, 1)$ and, in this case, for every $T > 0$,

$$\mathcal{P}_T(\omega, \cdot) := \mathcal{L}((B_t^{H, T})_{t \leq 0} | (B_t^H)_{t \leq 0} = (\omega_t)_{t \leq 0})$$

is a Feller transition on $\mathcal{W}_{\theta,\delta}(\mathbb{R}^-)$ where $B_t^{H,T} = B_{t+T}^H - B_T^H$. Let us now prove that $X^{(\infty,\gamma)}$ is adapted. It is enough to show that for every $0 \leq t \leq T$, for every bounded continuous functionals $f : \bar{\mathcal{C}}^\theta([0, t], \mathbb{R}^d) \rightarrow \mathbb{R}$, $g : \mathcal{W}_{\theta,\delta}((-\infty, T]) \rightarrow \mathbb{R}$ and $h : \mathcal{W}_{\theta,\delta}(\mathbb{R}_-) \rightarrow \mathbb{R}$

$$\begin{aligned} \mathbb{E}[f(X_s^{(\infty,\gamma)}, s \in [0, t])g(B_s^H, s \leq T)h(B_{s+t}^H, s \leq 0)] \\ = \mathbb{E}[f(X_s^{(\infty,\gamma)}, s \in [0, t])\psi^g(B_s^H, s \leq 0)h(B_{s+t}^H, s \leq 0)] \end{aligned} \quad (36)$$

where $\psi^g((\omega_s)_{s \leq t}) = \mathbb{E}[g(B_s^H, s \leq T) | (B_s^H)_{s \leq t} = (\omega_s)_{s \leq 0}]$. Owing to the Feller property on $\mathcal{P}_T(\omega, \cdot)$, we easily obtain that ψ^g is continuous on $\mathcal{W}_{\theta,\delta}(\mathbb{R}_-)$.

Then, using the ergodicity of the increments of B^H , we can show as in the beginning of the proof that $(\tilde{\mathcal{P}}^{(n,\gamma)}(\omega))_{n \geq 1}$ is tight on $\bar{\mathcal{C}}^\theta(\mathbb{R}_+, \mathbb{R}^d) \times \mathcal{W}_{\theta,\delta}(\mathbb{R})$. Thus, there exists *a.s.* a sequence (n_k) such that

$$\mathbb{E}[f(X_s^{(\infty,\gamma)}, s \leq t)g(B_s^H, s \geq T)h(B_s^H, s \leq t)] = \lim_{k \rightarrow +\infty} \frac{1}{n_k} \sum_{k=1}^{n_k} H_{k-1} J_k$$

$$\mathbb{E}[f(X_s^{(\infty,\gamma)}, s \leq t)\psi_t^g(B_s^H, s \leq t)h(B_s^H, s \leq t)] = \lim_{k \rightarrow +\infty} \frac{1}{n_k} \sum_{k=1}^{n_k} H_{k-1} \mathbb{E}[J_k | \mathcal{F}_{\gamma(k-1)+t}]$$

with $\mathcal{F}_t = \sigma(B_s^H, s \leq t)$, $H_k = f(\bar{X}_{\gamma k+s}^\gamma, s \leq t)h(B_{\gamma(k-1)+s+t}^H - B_{\gamma(k-1)}^H, s \leq 0)$, and $J_k = g(B_{\gamma(k-1)+s}^H - B_{\gamma(k-1)}^H, s \leq T)$. This implies that it is now enough to prove that

$$\frac{1}{n} \sum_{k=1}^n H_{k-1} (J_k - \mathbb{E}[J_k | \mathcal{F}_{\gamma(k-1)+t}]) \xrightarrow{n \rightarrow +\infty} 0 \quad a.s.$$

This point follows from a decomposition of the above sum in martingale increments and from classical martingale arguments (see proof of Proposition 6 of [3] for a similar proof). \square

5.2 Identification of limits when $\gamma \rightarrow 0^+$

In this part we fix a H -fractional Brownian motion B^H on $\bar{\mathcal{C}}^\theta(\mathbb{R}, \mathbb{R}^q)$ and we consider a pair $(X^{\infty,\gamma}, B^H)$ on $\bar{\mathcal{C}}^\theta(\mathbb{R}_+, \mathbb{R}^d) \times \bar{\mathcal{C}}^\theta(\mathbb{R}, \mathbb{R}^q)$ such that for each $\gamma > 0$ the joint distribution is given by Proposition 5.

PROPOSITION 6. *Let (γ_k) be a sequence converging to 0 such that the distributions of $(X^{\infty,\gamma_k}, B^H)$ are converging weakly on $\bar{\mathcal{C}}^\theta(\mathbb{R}_+, \mathbb{R}^d) \times \bar{\mathcal{C}}^\theta(\mathbb{R}, \mathbb{R}^q)$ to (X^∞, B^H) . Then the pair (X^∞, B^H) is a stationary adapted solution to (1) in the sense of Definition 1.*

Proof. Let us first introduce

$$\tilde{\Phi}(\alpha, \beta)_t := \alpha_0 + \int_0^t b(\tilde{\Phi}(\alpha, \beta)_s) ds + \int_0^t \sigma(\tilde{\Phi}(\alpha, \beta)_s) d\beta_s,$$

and remark that $\tilde{\Phi}(\alpha, \beta) = \Phi(a, \beta)$, if $\alpha_0 = a$. We want to show that

$$X^\infty = \tilde{\Phi}(X^\infty, B^H) \quad (37)$$

almost surely so that (X^∞, B^H) is a solution to (1). Let us rewrite the equation with the help of two continuous operators on $\bar{\mathcal{C}}^\theta(\mathbb{R}_+, \mathbb{R}^d) \times \bar{\mathcal{C}}^\theta(\mathbb{R}, \mathbb{R}^q)$:

$$\Psi(\alpha, \beta)_t = \int_0^t b(\alpha_s) ds + \int_0^t \sigma(\alpha_s) d\beta_s,$$

and

$$\Delta(\alpha)_t = \alpha_t - \alpha_0.$$

Then equation (37) is equivalent to

$$\Delta(X^\infty) = \Psi(X^\infty, B^H). \quad (38)$$

Let us also consider the discretization of Ψ

$$\Psi^\gamma(\alpha, \beta)_t = \int_0^t b(\alpha_{\underline{s}_\gamma}) ds + \int_0^t \sigma(\alpha_{\underline{s}_\gamma}) d\beta_s.$$

Obviously (6) can be rewritten

$$\Delta(X^{\infty, \gamma}) = \Psi^\gamma(X^{\infty, \gamma}, B^H). \quad (39)$$

LEMMA 4. *Let $(\gamma_k)_{k \geq 1}$ be a sequence converging to 0 such that $(X^{\infty, \gamma_k}, B^H)_{k \geq 1}$ converges weakly on $\bar{C}^\theta(\mathbb{R}_+, \mathbb{R}^d) \times \bar{C}^\theta(\mathbb{R}, \mathbb{R}^q)$ to (X^∞, B^H) . Then $\Psi^{\gamma_k}(X^{\infty, \gamma_k}, B^H)$ converges weakly on $\bar{C}^\theta(\mathbb{R}_+, \mathbb{R}^d)$ to $\Psi(X^\infty, B^H)$.*

Proof. Let $(\alpha, \beta) \in \bar{C}^\theta(\mathbb{R}_+, \mathbb{R}^d) \times \bar{C}^\theta(\mathbb{R}_+, \mathbb{R}^q)$. A classical result concerning the discretization of Young integrals shows that

$$|\Psi(\alpha, \beta)_t - \Psi^\gamma(\alpha, \beta)_t| \leq \|\alpha\|_{\theta, t} \|\beta\|_{\theta, t} \gamma^{2\theta-1} t.$$

See for instance [4], Proposition 31 or [30]. Hence for $T > 0$,

$$\|\Psi(\alpha, \beta) - \Psi^\gamma(\alpha, \beta)\|_{\theta, T} \leq \|\alpha\|_{\theta, T} \|\beta\|_{\theta, T} \gamma^{2\theta-1} T^{1-\theta}. \quad (40)$$

Let F be any bounded K -Lipschitz functional on $\bar{C}^\theta(\mathbb{R}_+, [0, T])$,

$$|\mathbb{E}(F(\Psi(X^{\infty, \gamma_k}, B^H))) - \mathbb{E}(F(\Psi(X^\infty, B^H)))| \rightarrow 0 \quad (41)$$

as $k \rightarrow \infty$. Then

$$|\mathbb{E}(F(\Psi^{\gamma_k}(X^{\infty, \gamma_k}, B^H))) - \mathbb{E}(F(\Psi(X^{\infty, \gamma_k}, B^H)))| \leq K \mathbb{E}(\|X^{\infty, \gamma_k}\|_{\theta, T} \|B^H\|_{\theta, T}) T^{1-\theta} \gamma_k^{2\theta-1}, \quad (42)$$

and using Proposition 3(ii) the left hand side of (42) is converging to 0 as $k \rightarrow \infty$. Combining (41) and this last fact, we get the desired convergence in distribution. \square

Let us start with

$$\Delta(X^{\infty, \gamma_k}) = \Psi^\gamma(X^{\infty, \gamma}, B^H), \quad (43)$$

and let $k \rightarrow \infty$. By Lemma 4, the right hand side of (43) converges to $\Psi(X^\infty, B^H)$ and the left hand side to $\Delta(X^\infty)$, which, in turn, has the same distribution as $\Psi(X^\infty, B^H)$.

Now, let us prove that X^∞ is stationary. It is enough to show that $\mathbb{E}[F(X^\infty)] = \mathbb{E}[F(X_{t+}^\infty)]$ for every $t \geq 0$ and for every functional F defined by $F(\alpha) = \prod_{i=1}^m f_i(\alpha_{t_i})$ where f_1, \dots, f_m denote Lipschitz continuous functions on \mathbb{R}^d and t_1, \dots, t_m belong to \mathbb{R}_+ . By Proposition 5, the distribution of $X^{\infty, \gamma}$ is invariant by the time-shift $(\theta_k \gamma)$ for every $k \in \mathbb{N}$ so that $\mathbb{E}[F(X^\infty)] = \mathbb{E}[F(X_{t+}^\infty)]$. The result follows easily by checking that for every $T > 0$,

$$\mathbb{E} \left[\sup_{u, v \in [0, T], |u-v| \leq \gamma} |X_v^{\infty, \gamma} - X_u^{\infty, \gamma}| \right] \xrightarrow{\gamma \rightarrow 0} 0.$$

Finally, it remains to show that (X^∞, B^H) is adapted. Since (X^{∞, γ_k}) converges in distribution to X^∞ on $\bar{C}^\theta(\mathbb{R}_+, \mathbb{R}^d)$ and since B^H belongs to $\mathcal{W}_{\theta, \delta}$ (with $\theta \in (1/2, H)$ and $\theta + \delta \in (H, 1)$), $(X^{\infty, \gamma'_k}, B^H)$ converges to (X^∞, B^H) for γ'_k a subsequence of γ_k . Then, we can let γ go to 0 in equality (36) and the result follows. \square

6 Simulations

In this section, we give an illustration of the application of our procedure for a one-dimensional toy equation:

$$dX_t = -X_t dt + (4 + \cos(X_t)) dB_t^H.$$

We propose to compute an estimation of the density of the (marginal) invariant distribution in this case. We denote it by ν_0^H . By Theorem 1, for every bounded continuous function $f : \mathbb{R}^d \rightarrow \mathbb{R}$,

$$\lim_{\gamma \rightarrow 0} \lim_{n \rightarrow +\infty} \mathcal{P}_0^{(n,\gamma)}(\omega, f) = \nu_0^H(f).$$

The first step is to simulate the sequence $(B_{\gamma k}^H - B_{\gamma(k-1)}^H)_{k=1}^n$. We use the method of Wood-Chan (see [29]) which is based on the embedding of the covariance matrix of the fractional increments in a symmetric circulant matrix (whose eigenvalues can be computed using the Fast Fourier Transform).

Then, we compute $K_h * \mathcal{P}_0^{(n,\gamma)}$ where K_h is the Gaussian convolution kernel defined by $K_h(x) = \frac{1}{\sqrt{2\pi}h} \exp(-\frac{x^2}{2h})$. Note that $K_h * \mathcal{P}_0^{(n,\gamma)}(x_0) = \mathcal{P}_0^{(n,\gamma)}(K_h(x_0 - \cdot))$, where, for a measure μ , and a μ -measurable function f , we set $\mu(f) = \int f d\mu$. In Figure 1 is depicted the approximate density with the following choices of parameters

$$n = 10^7, \quad \gamma = 0.05 \quad h = 0.2, \quad H = \frac{3}{4}.$$

We choose to compare it with the density of the invariant distribution when $H = 1/2$. Note that in this case, the invariant distribution is (semi)-explicit (as for every ergodic one-dimensional diffusion) and is given by

$$\nu_0^{\frac{1}{2}}(dx) = \frac{M(dx)}{M(\mathbb{R})} \quad \text{where} \quad M(dx) = \frac{1}{(4 + \cos x)^2} \exp\left(-\int_0^x \frac{2u}{(4 + \cos u)^2} du\right) dx.$$

We observe that the distribution when $H = 3/4$ has heavier tails than in the diffusion case. Finally, in order to have a rough idea of the rate of convergence, we depict in Figure 2 the

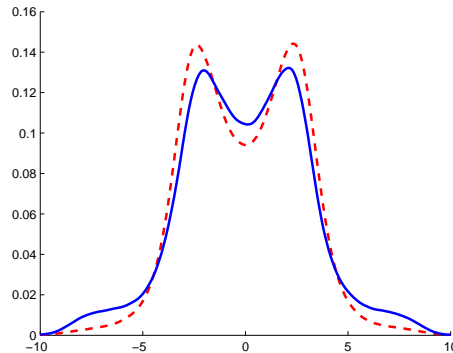


Figure 1: Approximate density of ν_0^H (continuous line) compared with that of $\nu_0^{\frac{1}{2}}$ (dotted line)

approximate densities for different values of n keeping the other parameters unchanged.

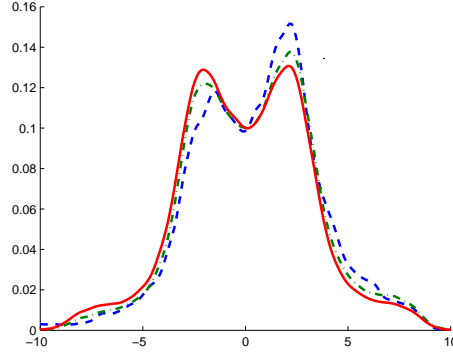


Figure 2: Approximate density of ν_0^H for $n = 10^5$ (dotted line), $n = 10^6$ (dash-dotted line), $n = 10^7$ (continuous line)

REMARK 5. As mentioned before, this section is only an illustration. In fact, there are (many) numerical open questions. For the estimation of the error, it would be necessary for a function f to get some rate of convergence results for $\mathcal{P}_0^{(n,\gamma)}(f) - \nu_H(f)$ (long-time error) and for $\nu_0^{H,\gamma}(f) - \nu_0^H(f)$ (discretization error) where $\nu_0^{H,\gamma}$ denotes the initial distribution of the stationary Euler scheme with step γ . Note that in the diffusion case, the long time error is managed by a CLT with rate $(\gamma n)^{-\frac{1}{2}}$ whereas the discretization error is $O(\gamma)$ (see [28]). Finally, even if the Wood and Chan simulation method is fast and exact, it requires a lot of memory because of the Fast Fourier Transform. On Matlab, for instance, this implies that we can not take n greater than 2.10^7 . Thus, it could be interesting to study some discretization schemes based on some approximations of the fBm-increments simulated, which consumes less memory.

7 Appendix

Proof of Proposition 1 Let us show that $(\bar{X}_{\gamma k})$ is a *skew-product* in the sense of [13] as follows. For a fractional Brownian B^H motion on \mathbb{R} , set for every $n \in \mathbb{Z}$ $\Delta_n^\gamma = B_{(n+1)\gamma}^H - B_{n\gamma}^H$. Setting $\mathcal{W} := (\mathbb{R}^d)^{\mathbb{Z}^-}$, we then introduce the regular conditional probability $\bar{\mathcal{P}}^\gamma : \mathcal{W} \rightarrow \mathcal{M}_1(\mathbb{R}^d)$ defined by¹:

$$\bar{\mathcal{P}}^\gamma(\omega) = \mathcal{L}(\Delta_1^\gamma | (\Delta_k^\gamma)_{k \leq 0} = \omega)$$

and denote by \mathcal{P}^γ the Feller transition on \mathcal{W} defined for every measurable function $f : \mathcal{W} \rightarrow \mathbb{R}$ by $\mathcal{P}^\gamma f(\omega) = \int_{\mathbb{R}^d} f(\omega \sqcup \tilde{\omega}) \bar{\mathcal{P}}^\gamma(\omega, d\tilde{\omega})$ where for $\omega \in (\mathbb{R}^d)^{\mathbb{Z}^-}$ and $\tilde{\omega} \in \mathbb{R}^d$, $\omega \sqcup \tilde{\omega} = (\dots, \omega_2, \omega_1, \omega_0, \tilde{\omega})$. Setting $\Phi^\gamma(x, \tilde{\omega}) = x + \gamma b(x) + \sigma(x)\tilde{\omega}$ and $\mathbb{P}_H^\gamma := \mathcal{L}((\Delta_n)_{n \leq 0})$, we have defined a skew-product $(\mathcal{W}, \mathbb{P}_H^\gamma, \mathcal{P}^\gamma, \mathbb{R}^d, \Phi^\gamma)$ with the transition operator \mathcal{Q}^γ on $\mathbb{R}^d \times \mathcal{W}$ defined by

$$\mathcal{Q}^\gamma f(x, \omega) = \int f(\Phi^\gamma(x, \omega')) \mathcal{P}^\gamma(\omega, d\omega'),$$

which describes the dynamics of the Euler scheme.

Then, thanks to Theorem 1.4.17 of [13], uniqueness of the adapted and stationary discrete Euler scheme $(\bar{X}_{\gamma k})$ (in distribution) holds, if the skew-product $(\mathcal{W}, \mathbb{P}_H^\gamma, \mathcal{P}^\gamma, \mathbb{R}^d, \Phi^\gamma)$ is

¹ Note that since $(\Delta_n^\gamma)_{n \in \mathbb{Z}}$ is a stationary sequence, $\mathcal{L}(\Delta_1^\gamma | (\Delta_k^\gamma)_{k \leq 0} = \omega) = \mathcal{L}(\Delta_{n+1}^\gamma | (\Delta_{n+k}^\gamma)_{k \leq 0} = \omega)$ for every $n \in \mathbb{Z}$.

strong Feller and topologically irreducible (in the sense of Definition 1.4.6 and 1.4.7 of [13]).

First, write $\tilde{\omega} = (\tilde{\omega}^1, \dots, \tilde{\omega}^q)$ and $\Phi^\gamma = (\Phi_1^\gamma, \dots, \Phi_d^\gamma)$. Denote by $M^\Phi(x, \tilde{\omega})$ the (discrete) Malliavin covariance matrix of Φ defined by

$$\forall (x, \tilde{\omega}) \in \mathbb{R}^d \times \mathbb{R}^d \quad \text{and } (i, j) \in \{1, \dots, d\}^2, \quad M_{i,j}^\Phi(x, \tilde{\omega}) := \sum_{k=1}^d \partial_{\tilde{\omega}^k} \Phi_i^\gamma(x, \tilde{\omega}) \partial_{\tilde{\omega}^k} \Phi_j^\gamma(x, \tilde{\omega}).$$

Thus, $M^\Phi(x, \tilde{\omega}) = (\sigma\sigma^*)(x)$ and since σ^{-1} is bounded (and continuous), it follows that $x \rightarrow (\det(M^\Phi)^{-1}(x, \omega))$ is bounded continuous. Second, the functions $D_\omega \Phi$, $D_\omega D_x \Phi$ and $D_\omega^2 \Phi$ are clearly bounded continuous. Finally, the sequence $((\Delta_n^\gamma)^1)$ has a spectral density f that satisfies $\int_{-\pi}^\pi (f(x))^{-1} dx < +\infty$ (see *e.g.* [2] for an explicit expression of f). Thus, it follows from Theorem 1.5.9 of [13] that the skew-product is strong Feller.

For the topological irreducibility, it is enough to show that for every $(x, \omega) \in \mathbb{R}^d \times \mathcal{W}$, for every $(y, \varepsilon) \in \mathbb{R}^d \times \mathbb{R}_+^*$, $\mathcal{Q}(x, \omega, B(y, \varepsilon) \times \mathcal{W}) > 0$. Since σ is invertible, the map Φ is controllable in the following sense: $\Phi(x, \tilde{\omega}_x) = y$ has a (unique) solution $\tilde{\omega} \in \mathbb{R}^q$, for every $x, y \in \mathbb{R}^d$. Furthermore, b and σ being continuous, for every $\varepsilon > 0$, there exists r_ε such that for every $\tilde{\omega} \in B(\tilde{\omega}_x, r_\varepsilon)$, $\Phi(x, \tilde{\omega}) \in B(y, \varepsilon)$. Thus,

$$\mathcal{Q}(x, \omega, B(y, \varepsilon) \times \mathcal{W}) \geq \bar{\mathcal{P}}(\omega, B(\tilde{\omega}_x, r_\varepsilon)) > 0$$

since $\bar{\mathcal{P}}(\omega, \cdot)$ is Gaussian with positive variance. This concludes the proof.

References

- [1] Ludwig Arnold. *Random dynamical systems*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998.
- [2] Jan Beran. *Statistics for long-memory processes*, volume 61 of *Monographs on Statistics and Applied Probability*. Chapman and Hall, New York, 1994.
- [3] Serge Cohen and Fabien Panloup. Approximation of stationary solutions of Gaussian driven stochastic differential equations. *Stochastic Process. Appl.*, 121(12):2776–2801, 2011.
- [4] Laure Coutin. Rough paths via sewing lemma. To appear in ESAIM PS, 2012.
- [5] Hans Crauel. Non-Markovian invariant measures are hyperbolic. *Stochastic Process. Appl.*, 45(1):13–28, 1993.
- [6] A. M. Davie. Differential equations driven by rough paths: an approach via discrete approximation. *Appl. Math. Res. Express. AMRX*, (2):Art. ID abm009, 40, 2007.
- [7] A. Deya, A. Neuenkirch, and S. Tindel. A Milstein-type scheme without Lévy area terms for SDEs driven by fractional Brownian motion. *Ann. Inst. Henri Poincaré Probab. Stat.*, 48(2):518–550, 2012.
- [8] Denis Feyel and Arnaud de La Pradelle. Curvilinear integrals along enriched paths. *Electron. J. Probab.*, 11:no. 34, 860–892 (electronic), 2006.
- [9] Peter K. Friz and Nicolas B. Victoir. *Multidimensional stochastic processes as rough paths*, volume 120 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2010. Theory and applications.

- [10] María J. Garrido-Atienza, Peter E. Kloeden, and Andreas Neuenkirch. Discretization of stationary solutions of stochastic systems driven by fractional Brownian motion. *Appl. Math. Optim.*, 60(2):151–172, 2009.
- [11] Paolo Guasoni. No arbitrage under transaction costs, with fractional Brownian motion and beyond. *Math. Finance*, 16(3):569–582, 2006.
- [12] Martin Hairer. Ergodicity of stochastic differential equations driven by fractional Brownian motion. *Ann. Probab.*, 33(2):703–758, 2005.
- [13] Martin Hairer. Ergodic properties of a class of non-Markovian processes. In *Trends in stochastic analysis*, volume 353 of *London Math. Soc. Lecture Note Ser.*, pages 65–98. Cambridge Univ. Press, Cambridge, 2009.
- [14] Martin Hairer and Alberto Ohashi. Ergodic theory for SDEs with extrinsic memory. *Ann. Probab.*, 35(5):1950–1977, 2007.
- [15] Jae-Hyung Jeon, Vincent Tejedor, Stas Burov, Eli Barkai, Christine Selhuber-Unkel, Kirstine Berg-Sørensen, Lene Oddershede, and Ralf Metzler. *In Vivo* anomalous diffusion and weak ergodicity breaking of lipid granules. *Phys. Rev. Lett.*, 106:048103, Jan 2011.
- [16] S. C. Kou. Stochastic modeling in nanoscale biophysics: subdiffusion within proteins. *Ann. Appl. Stat.*, 2(2):501–535, 2008.
- [17] Damien Lamberton and Gilles Pagès. Recursive computation of the invariant distribution of a diffusion. *Bernoulli*, 8(3):367–405, 2002.
- [18] Damien Lamberton and Gilles Pagès. Recursive computation of the invariant distribution of a diffusion: the case of a weakly mean reverting drift. *Stoch. Dyn.*, 3(4):435–451, 2003.
- [19] Vincent Lemaire. An adaptive scheme for the approximation of dissipative systems. *Stochastic Process. Appl.*, 117(10):1491–1518, 2007.
- [20] Terry Lyons. Differential equations driven by rough signals. I. An extension of an inequality of L. C. Young. *Math. Res. Lett.*, 1(4):451–464, 1994.
- [21] Benoit B. Mandelbrot and John W. Van Ness. Fractional Brownian motions, fractional noises and applications. *SIAM Rev.*, 10:422–437, 1968.
- [22] Gisirō Maruyama. The harmonic analysis of stationary stochastic processes. *Mem. Fac. Sci. Kyūsyū Univ. A.*, 4:45–106, 1949.
- [23] David Nualart and Aurel Răşcanu. Differential equations driven by fractional Brownian motion. *Collect. Math.*, 53(1):55–81, 2002.
- [24] David J. Odde, Elly M. Tanaka, Stacy S. Hawkins, and Helen M. Buettner. Stochastic dynamics of the nerve growth cone and its microtubules during neurite outgrowth. *Biotechnology and Bioengineering*, 50(4):452–461, 1996.
- [25] Gilles Pagès and Fabien Panloup. Approximation of the distribution of a stationary Markov process with application to option pricing. *Bernoulli*, 15(1):146–177, 2009.
- [26] Fabien Panloup. Recursive computation of the invariant measure of a stochastic differential equation driven by a Lévy process. *Ann. Appl. Probab.*, 18(2):379–426, 2008.

- [27] Alfredas Račkauskas and Charles Suquet. Central limit theorem in Hölder spaces. *Probab. Math. Statist.*, 19(1, Acta Univ. Wratislav. No. 2138):133–152, 1999.
- [28] Denis Talay. Second order discretization schemes of stochastic differential systems for the computation of the invariant law. *Stoch. Stoch. Rep.*, 29(1):13–35, 1990.
- [29] Andrew T. A. Wood and Grace Chan. Simulation of stationary Gaussian processes in $[0, 1]^d$. *J. Comput. Graph. Statist.*, 3(4):409–432, 1994.
- [30] L. C. Young. An inequality of the Hölder type, connected with Stieltjes integration. *Acta Math.*, 67(1):251–282, 1936.